THE IMAGINARY POINTS OF GEOMETRY.

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1. Infinity and Imaginary Points. Geometry was at one time regarded as describing actual spacial relations in an absolute and incontrovertible manner. Many philosophers supposed that by a process of pure deduction one started from self-obvious logical facts and discovered the inner constitution of space. Popular interest in the discovery of the now classical non-Euclidean geometry has been due largely to the still bewildering novelty of the fact that at certain stages in the development of an axiomatic system, there is a feature of arbitrary choice.

The geometry of real finite points being assumed in accordance with Euclidean standards, certain alternatives are yet open. One of these alternatives is in regard to the conception of points at infinity. As is well known, we may consistently adopt any one of the following conventions and indeed other possibilities remain.

(*a*) All points are finitely accessible. In other words there are no points at infinity.

(*b*) The points at infinity for a space of n dimensions constitute a (projective) space of $n - 1$ dimensions. Thus the Euclidean plane is completed by a line at infinity, the Euclidean three-space by a plane at infinity, and so forth. This is the convention of projective geometry.

(*c*) There is a single point at infinity. This is the convention of the geometry of inversion.

(*d*) The points at infinity fall upon n linear $(n - 1)$-spaces. Thus the Euclidean plane is completed by two intersecting straight lines at infinity, the Euclidean space by three planes at infinity and so forth. This is the convention of what has been called by some writers "function space."

The character of the set of points at infinity having been disposed of, the question of imaginary points is still open. The two common alternatives are as follows.

(*a*) To exclude from the discussion the consideration of imaginary points, or in other words to assume the existence of real points only.

(*b*) To assume the existence of points whose Cartesian coördinates are complex numbers as used in algebra.

So far as the writer is aware no other alternative has been currently suggested.

For problems of analysis situs and for investigations confined to a single net of rationality[1], it is assuredly permissible to assume the non-existence of imaginary points. In enumerative geometry, on the other hand, one has constant use for the so-called "fundamental law of algebra." The theorem that two curves, one of the mth order, and the other of the nth order, intersect in mn points not necessarily distinct, and like theorems, form the basis of this branch of mathe-

[1] Cf., for example, Veblen and Young, *Projective Geometry*, vol. 1, page 84.

matical science. For enumerative geometry, therefore, the convention that imaginary points exist with complex algebraic numbers for coördinates is essential.

But there are still other types of geometrical investigation and such as lead quite as naturally to a different method of extending the set of real points, and the present paper is intended to suggest some thoughts along this line.

Most mathematicians are familiar with the extensive use of complex numbers of the form $a + ib$, where $i^2 = -1$, in the study of electrical alternating current phenomena. A single algebraic variable is made to handle an essentially two-dimensional problem. In a similar manner, the methods of Hamilton are devised to treat a four-dimensional problem by the use of a single symbol, the quaternion, of the form $b = c_0 + ic_1 + jc_2 + kc_3$. The range of variation of two independent quarternions may be spoken of as a *two-space* or *plane*. A point in the plane will be given when particular values are assigned to the coordinates $(x_0 + ix_1 + jx_2 + kx_3,\ y_0 + iy_1 + jy_2 + ky_3)$. Here $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$, are eight real quantities in the ordinary sense.

The analysis in such a case differs somewhat from that for the usual complex numbers, since for quaternions multiplication is not in general commutative. Despite this fact one is occasionally obliged to investigate a problem in two quaternions, and the free use of geometric language serves here as elsewhere to render the results easily intelligible. This corresponds merely to speaking of a quaternion as a generalized number. A point in the plane will be called real, however, only if $x_1, x_2, x_3, y_1, y_2, y_3$ are simultaneously zero. The set of real points constitutes a real plane in the usual sense. The quaternion $b = c_0 + ic_1 + jc_2 + kc_3$ is from this point of view, real, when $c_1 = c_2 = c_3 = 0$, and "complex," or better, "hypercomplex" in other cases. The quaternions, $b = c_0 + ic_1 + jc_2 + kc_3$ and $\bar{b} = c_0 - ic_1 - jc_2 - kc_3$ are "conjugates," so that $b\bar{b} = \bar{b}b = c_0^2 + c_1^2 + c_2^2 + c_3^2$.

2. Vector Analysis. When we approach geometry from the standpoint of vector analysis, we find certain operations and rules of combination to be of great significance and others to be apparently of but slight importance. We shall illustrate our future remarks by reference to a space of three dimensions although all of our conclusions will be readily extended to linear spaces of any desired dimensionality.

Vectors are interpreted as points by the artifice of introducing a fixed origin, and of regarding any point as the terminal point of a vector originating from this fixed origin. Vectors and hence points are capable of unrestricted combination by addition and subtraction. Two vectors and hence two points, say a and a', determine a scalar called their inner product $a \cdot a'$. Any point may be multiplied by any scalar. Scalars in the ordinary theory are merely numbers and are called scalars to emphasize their distinction from vectors. Two or more points have furthermore an outer product that may be expressed by a determinant, when the given points are referred to any system of Cartesian coördinates having the given origin. The square of the distance between two points a and a' is a scalar expressible as $(a - a') \cdot \overline{(a - a')}$, where the second factor is the "conjugate"

of the first. Hence the point a' lies on the surface of a sphere of center a and radius r, if $(a - a') \cdot \overline{(a - a')} = r^2$. A fundamental property of vectors and hence of points in this system, is the following. If a and a' be any two points, $\sqrt{(a \cdot \bar{a})} + \sqrt{(a' \cdot \bar{a}')} \geqq \sqrt{[(a - a') \cdot \overline{(a - a')}]}$. In other words the sum of the lengths of two sides of a triangle is always greater than the length of the third side. It is to be noted that this inequality serves to give a meaning to certain infinite processes by making available a definition of limits among vectors.[1]

In extending a real system so as to include also imaginary points, a student versed in vector analysis is inclined to insist upon the preservation of such vectorial properties only as have proved most significant. To be sure, simple examples will always arise in which the imaginary points may be dispensed with. But when problems occur analogous to those with which vector analysis usually deals it is always a matter of interest to determine whether the customary methods are applicable or whether special technique must be developed. The outer product of two vectors is not commutative, and to sacrifice the commutative character of the inner product is not out of the question.

In the generalization to be considered, a point in three-space will continue to be designated by an ordered set of three non-homogeneous scalar coördinates (a_1, a_2, a_3). By this means the notation current for real points is preserved in the generalization, and the difficulties are referred a step further back, namely to the notion of scalar. Each coördinate is called a scalar or *generalized number* and is such as to reduce in a special case to a real number. We shall represent each of these scalars, a, in terms of more elementary quantities, b, in the following manner: Each a is a square matrix of order m, having the elements of the principal diagonal equal to each other, and elements below this principal diagonal, throughout equal to zero, while elements above this principal diagonal are left unrestricted. Any two scalars, a_1 and a_2, will be combined for purposes of addition and multiplication in accordance with the usual rules for adding and multiplying matrices, the multiplication for a pair of these square matrices being by rows of the first and columns of the second factor. The product of two generalized numbers is again a generalized number, but this product is not commutative unless further restrictions be imposed. Explicitly, let

$$a \equiv \begin{vmatrix} b_0, & b_{12}, & b_{13}, & \cdots, & b_{1m} \\ 0, & b_0, & b_{23}, & \cdots, & b_{2m} \\ 0, & 0, & b_0, & \cdots, & b_{3m} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & 0, & \cdots, & b_0 \end{vmatrix}.$$

The scalar, a, will be said to be "real" when and only when all of the elements above the principal diagonal vanish and when furthermore the single remaining independent element, namely the common element in the principal diagonal, is

[1] The importance of an inequality such as here illustrated for the establishment of limits, is well developed by J. Kürschák, "Über Limesbildung und allgemeine Körpertheorie", *Journal für die reine und angewandte Mathematik*, vol. 142, 1913, pp. 211-253.

itself "real," as this term will be later defined. By the *conjugate* of a, i.e., $\bar{a}$, is meant the scalar $b_0\bar{b}_0/a$, and it is defined only for such scalars as have $b_0 \neq 0$.

We may use the term *absolute value*, in connection with a scalar, a, and denote it by the symbol, $|a|$. This we define in terms of simpler elements, putting $|a|$ equal to $|b_0|$, where b_0 is the common diagonal element of the matrix defining the scalar, a, as above. As may be verified presently, the absolute value of any element, b, is so defined that the absolute value of a sum is equal to or less than the sum of the absolute values, and the absolute value of a product is equal to the product of the absolute values. The same propositions, therefore, will hold for the absolute value as applied to scalars, a, in fact $|a|^2 = a\bar{a}$, whenever $\bar{a}$ is defined. The distance from the origin to the point (a_1, a_2, a_3) is defined as the square root of the sum of the squares of the absolute values of the several coordinates. This is also the length of the vector from the origin to the point discussed. The question of signs arises here, in connection with the square root, in precisely the same manner that it does for real points. More generally, the inner product of the two points (a_1, a_2, a_3) and (a_1', a_2', a_3') in this order is defined as the scalar $a_1a_1' + a_2a_2' + a_3a_3'$. The sum of two vectors being defined in the ordinary manner, we shall have as a general theorem that the length of the vector obtained as the sum of two given vectors is equal to or less than the sum of the lengths of the two component vectors.

3. Quaternions as Elements. It remains to identify the elements, b, in terms of known quantities. One possibility is to consider the b's as themselves ordinary real numbers. A more interesting possibility has been suggested already, namely, each b may be taken as the quaternion expressed in the usual form, $b = c_0 + ic_1 + jc_2 + kc_3$, where c_0, c_1, c_2, c_3, are all ordinary real numbers, called the "fundamental components" of the quaternion b. If in particular, a subset be considered where the coefficients of j and k are zero for each of the quaternions, b, in question, these quaternions become merely ordinary complex numbers of algebra, or at least may be so regarded. If further, the order, m, of the matrices a, be unity, the scalars, a, will be themselves ordinary complex numbers.

Equations in vector analysis with real coefficients may frequently be interpreted in terms of these generalized numbers and so refer as well to imaginary points of geometry, different from those usually treated. When m, the order of the square matrix used in defining the "scalar," a, is greater than unity, we are confronted with the difficulty that the conjugate of a sum is not in general the sum of the conjugates of the given quantities. Thus the rationalization of even so simple an equation as that of the spheroid whose bipolar equation may be written $\sqrt{[(x-a)\cdot\overline{(x-a)}]} + \sqrt{[(x-a')\cdot\overline{(x-a')}]} = \sqrt{(d\cdot\bar{d})}$ presents difficulties in the general case. Other inconvenient features are likely to impress any one more interested in the simplicity of handling equations than in the generality of the results secured. However, considerations of this sort are of interest, since, on the one hand, they show the possibilities of other interpretations of equations which have long been studied and, on the other hand, indicate the availability

of geometrical language in domains which might, at first, seem unsuited to such phraseology. If the order, m, of the matrices representing generalized numbers be allowed to become infinite in an appropriate manner, these generalized numbers become operators of the form used in the theory of integral equations. While it has not been customary to study integral equations in the domain of quaternions, some interesting results might be obtained in such a case.

LINES OF ILLUMINATION CAUSED BY THE PASSAGE OF LIGHT THROUGH A SCREEN.[1]

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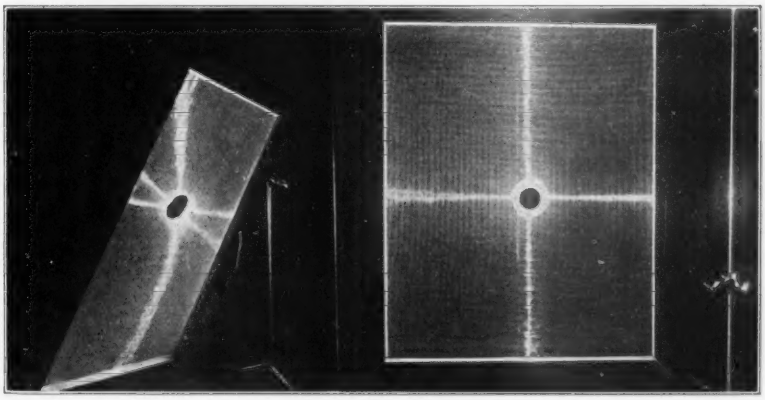
1. Description. On viewing a point source of light through a screen one will see on the screen several streamers of light emanating from the point in which the plane of the screen is pierced by the line connecting the eye with the light. The number of streamers which appear depends upon the direction of this line with respect to the plane of the screen. If, in particular, this line is perpendicular to the plane of the screen, there are four streamers, or two lines of light which cut at right angles. This is also the case for some other positions of the light and eye. In general, however, there are six streamers of light. On an old screen, these streamers are short and straight and one gets the impression of seeing a four- or six-pointed star. This is especially noticeable on looking at a source of light through the dust screen of a sleeping-car window. At first, when the source of light is far ahead of the train, the star will appear to have six points, but when the train is opposite the source of light the star appears to have only four points. If the screen is new and is made of polished wires of brass or copper, the streamers



extend to the frame and are well-defined curves. Views of this phenomenon are shown in the accompanying photographic reproductions. The black spot in these

[1] Presented to the American Mathematical Society, November 27, 1915. See *Bulletin* of that Society, vol. 22, p. 218.

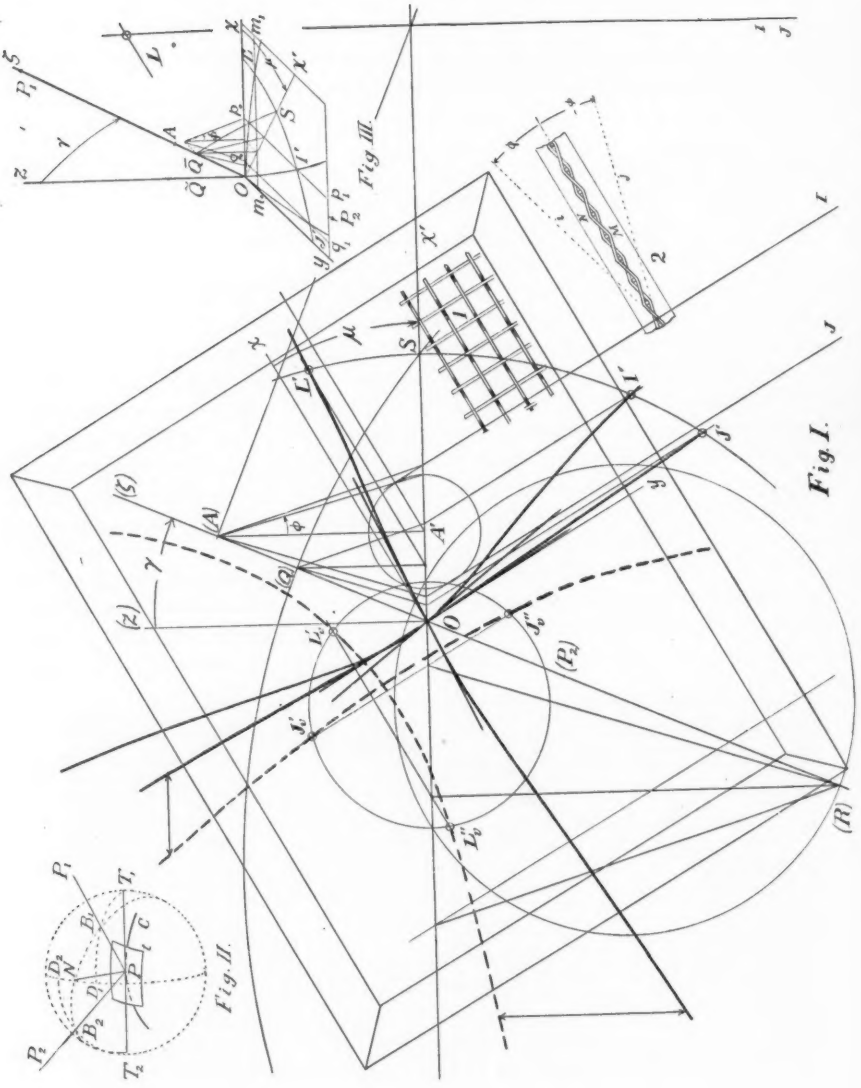


Fig. I.
Fig. II.
Fig. III.

reproductions is due to the presence of a small disk which was used to obscure the intense direct light from the source.

2. Cause. A close examination of the screen will show that the screening, from which it is made, must have been woven like cloth. The screening consists of a series of longitudinal wires, called the warp, which are interlaced with a series of tranverse wires, called the woof. In the weaving process of raising certain of the longitudinal wires and depressing others so as to form a *shed* for the passage of the transverse wires, the warp wires become crinkly, or wavy, but the woof wires remain unbent. The result of this operation is, that after the screening is attached to the frame, the wires of the woof are practically straight, parallel to each other and lie in the plane of the screen (or rather between two parallel planes which are separated by a distance equal to the diameter of these wires). The wires of the warp, however, are crinkly, or wavy, because they bend in and out around the straight wires of the woof, and each such wire lies in a plane perpendicular to the wires of the woof (or rather, between two such planes which are separated by a distance equal to the diameter of this wire). A clear notion of this construction may be obtained from views 1 and 2 of Fig. I. In view 2 the wires of the woof are represented by their cross sections, which are small circles separated by intervals of constant length, and the wires of the warp are shown as broad wavy lines. In view 1, the wires of the woof are represented by broad unshaded straight lines, while those of the warp are represented by broad straight lines which are shaded so as to represent the bending which takes place in crossing the wires of the woof.

From this description it is clear that the bend points of the wires of the warp, like M and N in view 2 of Fig. I, either receive no light from the source, which is on one side of the screen, or are obscured from the observer who is on the other side. Therefore, of the wires of the warp, only those approximately straight portions, which lie between the bend points, are effective in reflecting the light which comes from a source on one side of the screen to an observer who is situated on the other side. The straight wires of the woof (or, rather, those portions not obscured by the wires of the warp) are also effective in reflecting this light.

As far as its ability to reflect light from a source on one side to an observer on the other side is concerned, the screen may be regarded as composed of three groups of short, needlelike, pieces of wire. Those of one group (the portions of the wires of the woof not obscured by, or in the shadow of, the wires of the warp) lie in the plane of the screen and are parallel to each other. Those of each of the other two groups (composed of the short straight portions of the wires of the warp) are parallel to one another, are inclined to the plane of the screen at a constant angle and are perpendicular to the wires of the woof. View 2, Fig. I, represents the wires of one of these two groups as parallel to the line i and those of the other as parallel to the line j.

The wires of the screen, or rather the groups of short portions just described, reflect the light which falls upon them from the source, and many of the reflected rays reach the eye of the observer (even though it be on the side of the screen

opposite to that of the light). Those points of the wires where the rays are reflected which reach the eye of the observer, are the images of the source of light in the reflecting surface of the wires, and are called brilliant points. On account of the fineness of the mesh of the screen, consecutive brilliant points are close together, and it is the aggregate of these closely packed brilliant points which forms the streamers of light observed.[1]

3. Assumptions, Definitions, Tests. The solution of our problem may be much simplified without any appreciable loss in value, by assuming that the wires composing the screen are mathematical curves. For, the position of a brilliant point of a wire of small cross section differs very slightly from the position of the point which is approached by this brilliant point, as the cross section of the wire is diminished indefinitely. The geometric description of this limit point on the mathematical curve approached by the wire furnishes a definition for the brilliant point of a curve.

From the laws of the reflection of light it follows that the brilliant point of a surface may be described according to

DEFINITION 1. *A point P is said to be a brilliant point of a surface t with respect to a source of light P_1 and an observer's eye P_2, if the internal bisector of the angle P_1PP_2 is the normal to this surface at the point P.* See Fig. II.

If now the reflecting surface is supposed to be tubular[2] (like that of a wire) and the diameter of a cross section of this tube be regarded as infinitesimal (*i.e.*, as approaching zero), the brilliant point approaches a point on the mathematical curve approached by the tube which may be described by

DEFINITION 2. *A point P is said to be a brilliant point of a curve c with respect to a source of light P_1 and an observer's eye P_2, if the internal bisector of the angle P_1PP_2 is a normal to this curve at the point P.* See Fig. II.

It is of course evident that the rays PP_1 and PP_2 do not, in general, lie in the same tangent plane to the curve at the point P. If in definitions 1 and 2 the words "internal bisector" be replaced by the words "external bisector," these definitions become respectively those of the virtual brilliant point of a surface and a curve. Such points are not visible, but they present themselves in the analytic treatment when certain equations are freed from radicals.[3]

The method of treatment used in this paper was selected not only for the purpose of obtaining the equation of the brilliant point locus in question, but also

[1] At the suggestion of a referee, attention is here called to the fact that the spacing of the wires of the screen is so great as to make the phenomenon of diffraction negligible. The phenomenon described in this paper should therefore not be confused with the phenomenon observed in looking at a source of light through the stretched cover of an open umbrella.

[2] A tubular surface may be defined as the envelope of a sphere of constant radius which moves so that its center always lies on a curve (plane or skew), which is called the *axis*. If, in particular, the axis is a straight line, the surface becomes a cylinder of revolution; and if the axis is a circle, the surface is a torus.

[3] In definitions given by the author in the *Transactions of the American Mathematical Society*, vol. 9, pp. 245–279, the term *actual brilliant point* was used to designate the point which is here called simply *brilliant point*, and the term *brilliant point* was used to include both actual and *virtual brilliant points*.

with a view to furnishing a method of constructing this locus by means of circles and straight lines.[1]

In order to understand this method it will be necessary to know the following property of a sphere. If any point P on the surface of a sphere be connected by right line segments with each of a pair of points P_1 and P_2, which are conjugate relative to an inversion with respect to this sphere, and also with the points Q and R in which the diameter containing P_1 and P_2 pierces the sphere, the latter segments bisect internally and externally the angle formed by the former.

Any two points P_1 and P_2 of space constitute such a pair of points for any member of a certain one-parameter family of spheres. If r_1 and r_2 denote the distances of a general point P of space from the points P_1 and P_2, respectively, this family of spheres is represented by the equation

$$\frac{r_1}{r_2} = k, \tag{1}$$

in which k is a (positive) constant for any particular sphere of the family.

This property of the sphere furnishes the following

TEST. *To determine whether a point P of a curve C is a brilliant point of C with respect to two points P_1 and P_2, construct the normal plane to the curve at the point P and denote by Q the point in which this normal plane cuts the line P_1P_2. If Q lies between P_1 and P_2, pass through it a sphere having P_1 and P_2 as conjugate points. Then the point P is, or is not, a brilliant point of the curve C with respect to P_1 and P_2, according as it does, or does not, lie on the surface of this sphere. If, in particular, the normal plane contains both P_1 and P_2, the point P is a brilliant point.*[2]

The truth of the statement of this test lies in the fact that the normal line PQ to the curve C at the point P does, or does not, bisect internally the angle P_1PP_2 according as P does, or does not, lie on the sphere which passes through Q and has P_1 and P_2 as a pair of conjugate points. If, in particular, the normal plane contains both P_1 and P_2, the bisector of the angle P_1PP_2 is surely a normal to the curve at the point P.

4. Solution of the Problem. In order to find the equations of the streamers of light described in section 1, we will make use of the test just given. To this end we will choose (as in Fig. III) a set of rectangular axes, Ox, Oy, Oz, of which the origin O is the point where the line P_1P_2 (connecting the eye P_2 with the source of light P_1) pierces the plane of the screen, which is taken to be coincident with the plane xOy, and the axis Oy is coincident with the wire of the woof passing through the origin O. We will further choose on the line P_1P_2 an axis $O\zeta$ of which the origin is the point O and the positive sense is from P_2 (taken below

[1] For other methods of treatment the reader is referred to an article by the author entitled "Brilliant point phenomena" published in the *Washington University Studies*, vol. 8, Scientific Series, pp. 131–160, 1921.

[2] To get the corresponding test for a virtual brilliant point, it is only necessary to replace the words "lies between P_1 and P_2" by the words "lies beyond the limits of the segment P_1P_2." If, in particular, the normal plane is parallel to the line P_1P_2 the sphere becomes the perpendicular bisecting plane of the segment P_1P_2.

the screen) to P_1 (above the screen), and we will denote by ρ the (perpendicular) distance of a general point P from this axis. We will denote by A the point halfway between P_1 and P_2, and by h and c the distances OA and AP_1, respectively.

The quantities r_1 and r_2 of equation (1) may now be expressed by the formulas $r_1 = \sqrt{[(\zeta - h - c)^2 + \rho^2]}$, $r_2 = \sqrt{[(\zeta - h + c)^2 + \rho^2]}$ and the equation (1) thus becomes

$$\frac{(\zeta - h)^2 + \rho^2 + c^2}{\zeta - h} = 2l, \tag{2}$$

in which $l = c(1 + k^2)/(1 - k^2)$ is the distance of the center of this sphere from the point A.

To express the coördinates ρ and ζ in terms of x, y, z, we will denote by α, β, γ the direction angles of the axis $O\zeta$ with respect to the set of axes, Ox, Oy, Oz. Then, evidently,

$$\begin{aligned} \rho^2 + \zeta^2 &= x^2 + y^2 + z^2, \\ \zeta &= x\cos\alpha + y\cos\beta + z\cos\gamma. \end{aligned} \tag{3}$$

Making these substitutions, equation (2) becomes

$$\frac{x^2 + y^2 + z^2 - 2h(x\cos\alpha + y\cos\beta + z\cos\gamma) + h^2 + c^2}{x\cos\alpha + y\cos\beta + z\cos\gamma - h} = 2l. \tag{4}$$

By allowing l in this equation to take on all real values, we obtain all the spheres which have P_1 and P_2 as a pair of conjugate points.

We will denote by $\bar{Q}$ (Fig. III) the point in which the line P_1P_2 (*i.e.*, the axis $O\zeta$) is cut by a common normal plane of the wires of the woof. This plane has the equation

$$y = \bar{y} = \text{constant } (= Om_0 \text{ in Fig. III}), \tag{5}$$

and therefore $\bar{Q}$ is determined by the relations $\bar{\zeta} = O\bar{Q} = \bar{y}/\cos\beta$, $\bar{\rho} = 0$. Substituting these values for ζ and ρ in equation (2) we obtain the expression

$$2l = \left[\frac{\bar{y}^2}{\cos^2\beta} - 2h\frac{\bar{y}}{\cos\beta} + h^2 + c^2\right] \Big/ \left[\frac{\bar{y}}{\cos\beta} - h\right], \tag{6}$$

which gives the value of l for that sphere of the family (4) which passes through $\bar{Q}$. This sphere cuts the plane $zO\zeta$ in the circle $\bar{Q}S$ of Fig. III and the plane (xOy) of the screen in the circle $L'SI'J'$. The latter circle is represented by the equation which is obtained from equation (4) by giving $2l$ the value (6) and putting $z = 0$, *i.e.*, by the equation

$$\frac{x^2 + y^2 - 2h(x\cos\alpha + y\cos\beta) + h^2 + c^2}{x\cos\alpha + y\cos\beta - h} = \frac{\dfrac{\bar{y}^2}{\cos^2\beta} - 2h\dfrac{\bar{y}}{\cos\beta} + h^2 + c^2}{\dfrac{\bar{y}}{\cos\beta} - h}. \tag{7}$$

On the other hand, the plane (5) cuts the plane (xOy) of the screen in the line m_0m_1 (Fig. III), which is represented by the equations

$$y = \bar{y}, \qquad z = 0. \tag{8}$$

According to the Test (stated in the preceding section), those points only (like L' of Fig. III) of the line (8) [m_0m_1 of Fig. III] are brilliant points of wires of the woof, with respect to P_1 and P_2, in which this line is cut by the circle (7) [$L'SI'J'$ of Fig. III]. Consequently, the locus of brilliant points (actual and virtual) of the wires of the woof is obtained by giving $\bar{y}(= Om_0$ in Fig. III) all possible real values, *i.e.*, by eliminating $\bar{y}$ between equations (7) and (8). The resultant of this elimination may be written in the form:

$$xy^2 \cos\alpha - x^2 y \cos\beta + h(x^2\cos^2\beta - y^2\sin^2\beta) + (c^2 - h^2)\cos\beta(x\cos\alpha\cos\beta - y\sin^2\beta) = 0. \tag{9}$$

In order to determine the locus of brilliant points of the short (approximately) straight portions of the wires of the warp, we will denote by Q (Fig. III) the point in which the line P_1P_2 (*i.e.*, the axis $O\zeta$) is cut by a plane perpendicular to one, or other, of the two groups of such short portions. The equation of such a plane may be written in the form

$$\pm \epsilon z + x = \bar{x} = \left\{ \begin{matrix} Op_0 \\ Oq_0 \end{matrix} \text{ in Fig. III} \right\}, \tag{10}$$

in which $\bar{x}$ is the distance of the horizontal trace of this plane from the axis Oy, and $\epsilon = \tan\phi$ where ϕ is the angle of inclination of the short straight portions of the wires of the warp to the plane (xOy) of the screen. Then, for the upper sign equation (10) represents a plane perpendicular to the line i of view 2, Fig. I, and for the lower sign it represents a plane perpendicular to the line j. Let us denote the coördinates of $\tilde{Q}$ by $(\tilde{x}, \tilde{y}, \tilde{z})$. Since $\tilde{Q}$ is a point of the plane (10), we have $(\pm\epsilon\tilde{z} + \tilde{x})/\tilde{\zeta} = \bar{x}/\tilde{\zeta}$ or $\pm\epsilon\cos\gamma + \cos\alpha = \bar{x}/\tilde{\zeta}$, whence $\tilde{\zeta} = O\tilde{Q} = \bar{x}/e$, $\tilde{\rho} = 0$, where

$$e = \cos\alpha \pm \epsilon\cos\gamma. \tag{11}$$

Putting these values for ζ and ρ in equation (2), we obtain the relation

$$2l = \left[\frac{\bar{x}^2}{e^2} - 2h\frac{\bar{x}}{e} + h^2 + c^2\right] \Big/ \left[\frac{\bar{x}}{e} - h\right], \tag{12}$$

which gives the value of l for that sphere of the family (4) which passes through $\tilde{Q}$. This sphere cuts the plane $(z = 0)$ of the screen in the circle

$$\frac{x^2 + y^2 - 2h(x\cos\alpha + y\cos\beta) + h^2 + c^2}{x\cos\alpha + y\cos\beta - h} = \frac{\dfrac{\bar{x}^2}{e^2} - 2h\dfrac{\bar{x}}{e} + h^2 + c^2}{\dfrac{\bar{x}}{e} - h}. \tag{13}$$

It is no loss of generality to assume, as in Fig. III, that $\tilde{Q}$ coincides with $\bar{Q}$. Hence equation (13) is also represented by the circle $L'SI'J'$ in Fig. III. The plane (10) cuts the plane of the screen in the trace whose equations are

$$x = \bar{x}, \qquad z = 0. \tag{14}$$

In Fig. III this trace is represented by the line p_0p_1 or q_0q_1 according as the upper or lower sign of equation (10) is considered. According to the Test, those points only (like I' or J' of Fig. III) of the line (14) are brilliant points, with respect to P_1 and P_2, of wires of the warp (or rather, of the short straight portions thereof) in which this line is cut by the circle (13). Consequently, the locus of brilliant points (actual and virtual) of the short straight portions of the wires of the warp (and hence of the wires of the warp themselves, since, as we have seen, the bend points yield no brilliant points) may be obtained by giving $\bar{x}$ all possible real values, that is, by eliminating $\bar{x}$ between equations (13) and (14). The eliminant thus obtained may be written in the form

$$\pm \epsilon x^3 \cos\gamma - x^2y\cos\beta + exy^2 + h[(1-e^2)x^2 - e^2y^2] + e(c^2-h^2)[(1-e\cos\alpha)x - ey\cos\beta] = 0, \tag{15}$$

in which e stands for the expression (11). For the upper sign this equation represents the brilliant point locus for the short straight portions of the wires of the warp which are parallel to the line i of view 2, Fig. I, and for the lower sign it represents the corresponding locus for the portions parallel to the line j.

The space construction represented in (cavalier) perspective in Fig. III is executed in the drawing plane of Fig. I by means of the Mongean method of descriptive geometry; thus the graphs of the curves (9) and (15) are obtained in Fig. I. The curve $OL'L_v'L_v''$ is the graph of the equation (9), the curve $OJ'J_v'J_v''$ is the graph of equation (15) for the lower sign, and the curve $O'I'$ is the graph of equation (15) for the upper sign. The full portions of these curves are the loci of the (actual) visible brilliant points and the dashed portions are the loci of the invisible virtual brilliant points.

If we equate to zero the first degree terms of equations (9) and (15) we obtain the equations

$$(1-\cos^2\beta)y - \cos\alpha\cos\beta\cdot x = 0, \tag{16}$$

$$[1 - \cos\alpha(\cos\alpha \pm \epsilon\cos\gamma)]x - \cos\beta(\cos\alpha \pm \epsilon\cos\gamma)y = 0, \tag{17}$$

of the tangents to these curves at the origin. These are, in fact, the equations of the short straight streamers which we observed in viewing a light through a tarnished screen. From equation (17) it is evident that the two lines which this equation represents coalesce when $\beta = 90°$ (*i.e.*, when the line connecting the eye with the light is perpendicular to the wires of the woof). In this case equations (16) and (17) become $y = 0$ and $x = 0$ respectively. Thus for $\beta = 90°$ the six-pointed star becomes a four-pointed star.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

22. Sir David Brewster and the Stereoscope.

To students of mathematics Sir David Brewster (1781–1868) is best known for his *Life of Newton* (1828, greatly enlarged in 1855); to the physicist he is best known for his work in optics; and to the average citizen he is not known at all, although to him is due the kaleidoscope (1816), the major part of the invention of the stereoscope, and probably the first suggestions of the opera glass.

In connection with the invention of the stereoscope (1849), which had been to some slight degree anticipated by Wheatstone (1838), and which was greatly improved by our Dr. Oliver Wendell Holmes, he had a just grievance against his one-time friend, the Abbé Moigno (1804–1884), of Paris, a physicist and mathematician of considerable ability, and the founder of *Cosmos* (1852). Moigno had not given Brewster the credit which the latter felt was his due, and as a result a protest had been made. Moigno's failure to publish the correspondence in full in *Cosmos* resulted in a number of heated communications from Brewster, several (and probably all) of which are now in my collection. One of them, now published I believe for the first time, is given below and shows that the President of the Peace Congress, held five years earlier in London, was prepared for battle when he felt that his rights of discovery were not duly recognized.

The letter is as follows:

Dear Abbé Moigno,

I am very unwilling to do anything disagreeable to you, but *truth must be told* and *justice done* whatever be the cost.

By your delay in publishing a reply to an article in *Cosmos*, and your not having fulfilled the promise you lately made to me in Paris to publish it, you have driven me to write the enclosed pages which are still *unpublished and private.*

I will, of course, strike out the observations *which refer to you personally* if you publish, without comment, the communication I previously sent you.

Mr. Wheatstone, *who knows well that no stereoscope with prisms was ever made for me*, has *allowed you* to continue in the belief which you erroneously cherished, and has thus acted a most dishonest part.

The idea of a *Refracting Stereoscope* was *first* given by me, and *first* published by me in the article from the *Phil. Mag.* which I gave you in 1850.

I call your attention especially to the hidden, or rather palpable meaning of your phrase

Le croirait on?

and trust that as a man of honour you will do me justice. I am,

Ever Most Truly yrs.

D. Brewster

St. Leonards College
St. Andrews, May 8th, 1856.

23. Clifford's Genius Shown as a Boy.

Newton's remark that if Cotes had lived "we might have known something" may also be applied to the case of William Kingdon Clifford, one of the most promising of the British mathematicians of his day. Born at Exeter in 1845,

he died at Madeira, where he had gone in the vain hope of recovering his health, at the age of thirty-four. He was only twenty-six when he was made professor of applied mathematics at University College, London, and only twenty-nine when he was elected a Fellow of the Royal Society. His contributions to the study of the graphic methods of Möbius, to the theory of Riemann's surfaces, and to the development of biquaternions, together with his *Common Sense of the Exact Sciences*, are all well known, as is the edition of his *Mathematical Papers* by Mr. Tucker in 1882.

Among several of his autograph letters now in my collection is one of particular interest, written on St. Giles's Day (September 1) 1863, just before he entered Trinity College, Cambridge. It shows in a very personal manner the nature of his interests at an age when most boys are concerned with pursuits that are quite in contrast with those which he sets forth.

The letter is as follows:

9 Park Place, Hill's Court,
EXETER, S:Giles, 1863.

My dear Sir

I thank you very much indeed for your kindness, and am sorry that I should have given you the trouble to write to me. I have been on a walking tour in the North of Devon, or I should have written long before. By to-morrow I hope to send you something, and will do what I can to follow out your kind suggestions; but I am a very junior reader myself. I have had in my mind almost from the time I began to fly kites (I have not yet left off) the problem of finding the form of a kite-string under the action of the wind. On a rough trial the other day, the intrinsic equation seemed not very difficult to obtain; if I get at any result, I will send it you to-morrow. I have been trying to construct a second interpretation of mechanical equations, similar to that of tangential coordinates, but have failed hitherto. Being a firm believer in the duality of symbols, I should look upon complete failure as a proof that our symbolical system is wrong. You will be amused by my visionary attempt at obtaining a method of inventing problems by the dozen.

With best thanks, believe me to remain,

Yours very sincerely
+ W: K: CLIFFORD.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

A number of questions proposed at various times preceding the present year are reprinted below in the hope that fresh interest may be aroused in them. Some of these have already called forth replies, as indicated in the notes, and in several cases very profitable discussions have arisen; but the questions have still to be completely settled.

At the same time the editor hopes that there will continue to be a lively flow of discussion on new matters. The unfortunate delay in the appearance of recent numbers of the MONTHLY has probably interrupted to some extent the correspondence of this department; and the consequent speeding up in publication which is now looked for calls for a proportionate acceleration in the supply of contributed material. The function of these columns is to afford opportunity for the exchange of views of all possible types; and this end may be furthered by the number, variety, and suggestiveness of questions proposed, and by a liberal supply of answers and discussions on both old and new topics.

REPRINT OF OLD QUESTIONS.

15 [*1914*, 278; *1916*, 353; *1919*, 68; *1920*, 114, 361; *1921*, 124]. In the *Proceedings of the Royal Society of Edinburgh*, vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

Students are notoriously suspicious of those steps which an author announces as "easy," and are sometimes inclined to believe that the word is used in a humorous sense. The present case almost justifies such an attitude, since so far no reader has furnished a satisfactory explanation, and it is difficult to resist the suspicion that the "proof" is an illusion.

In a previous editorial note (*1920*, 361) a statement was made of the principal types of answer arriving in response to this question. To prevent the repetition of a former misunderstanding, it may be said here that the question is not how the second equation may be deduced from the first, but how the second equation may be used to facilitate the proof that the first is impossible. There are of course proofs in existence that the sum of two cubes cannot be a cube.[1]

21 [*1914*, 341; *1916*, 354; *1919*, 68, 239; *1920*, 114; *1921*, 114]. For the Diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$x =$	3,	4,	5,	9,	23,	282,	375,	378661,
$y =$	-2,	-1,	2,	4,	8,	43,	52,	5234.

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given Diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

A note by E. B. Escott (*1919*, 239) contained several methods for finding solutions, and references to the literature[2] suggested the probability that an infinite number of integral solutions existed; but no others were given. Hence both parts of the question await a final reply.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460; *1921*, 19, 125]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h) \right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

Although nine replies by six different writers have been published, the problem is by no means exhausted. As to the interpretation of the question, there is no problem worth discussing if h is taken to be constant, so that each correspondent has naturally assumed that the equation is to be true for a variable h over the interval $0 < h < H$. Starting from the well-known fact that the equation is true for polynomials of degree three at most,[3] E. Swift (*1920*, 301) proved that it was

[1] See, for instance, R. D. Carmichael, *Diophantine Analysis*, p. 67.

[2] L. J. Mordell, *Proceedings of London Mathematical Society*, series 2, vol. 13, 1914, pp. 60–80. See also L. E. Dickson, *History of the Theory of Numbers*, vol. 2, pp. 537–9.

[3] The prismatoid formula is that special case of Cotes' formula of approximate integration for which the interval is divided into *two* equal parts, and the integrand assumed to be of degree two. When the method is applied to a polynomial of degree three, the interpolation for the

true for no other analytic functions regular at the origin, and A. A. Bennett (*1920*, 301) obtained a somewhat more comprehensive result. D. C. Gillespie (*1920*, 405) showed that the equation was satisfied by no other function having six continuous derivatives in the interval. On the other hand, A. A. Bennett (*1920*, 460) found analytic solutions of the type $f(x) = x^2 \sin(b \log x - c)$ having essential singularities at the origin; and, in editorial remarks by W. A. Hurwitz (*1920*, 462), it was shown (1) that the value of b may be real, (2) that the function then has, from the real variable point of view, a continuous derivative at the origin. The main question now is: *How many continuous derivatives at most may $f(x)$ possess at the origin, if $f(x)$ satisfies the given equation, but is not of the form $ax^3 + bx^2 + cx + d$?* The answer may be anything from *one* to *five* inclusive.

Throughout the discussion the behavior of the function in the neighborhood of the origin is the point of critical importance, and the problem is fundamentally different when so modified that the identity of the origin is lost. Thus the related question where the equation is to be true after an arbitrary translation of axes was satisfactorily answered by J. P. Ballantine (*1921*, 19), who showed that $ax^3 + bx^2 + cx + d$ was the only *continuous* solution.[1]

36 [*1919*, 69, 291–295], part 1. For what values of n can $\cos 2\pi/n$ be expressed in the form $(a + \sqrt{b})/c$, where a, b, and c are integers?

The other parts of this question, relating to equations of the third and fourth degrees, have been answered (*1919*, 292). In a recent issue (*1921*, 374) R. S. Underwood answered a question very near to this, showing that, of angles commensurable with π, only multiples of $\pi/3$ have *rational* cosines. We know that the answer to the present question includes multiples of $\pi/4$, $\pi/5$ and $\pi/6$. What else does it include?

39 [*1920*, 256; *1921*, 125]. There are certain problems in geometry which are simple in statement but can be reduced only to very complicated problems in transcendental analysis. Following are several examples of the type of problem in question.

1. What is the smallest plane area within which a given figure can be turned through a complete revolution? It is not implied that the figure should revolve about a fixed point, but merely that in the course of its motion it should have every possible orientation in the plane. The problem may be modified by considering only convex areas.

An interesting special case is that in which the given figure is a segment of a straight line. In this case it has been conjectured by Professors Osgood and Kubota that the smallest area may

integrand is inexact, but the value of the definite integral is by accident correct. More generally, if $2n$ equal subdivisions are used, the definite integral will be found to be correctly evaluated, not only for an integrand of degree $2n$, but also for one of degree $2n + 1$. Gauss was perhaps the first to notice this fact (see *Encyclopédie des Sciences Mathématiques*, I 21, pp. 120–2). It is ignored in a number of the standard texts dealing with interpolation; though it is quite evident when central differences are used. In fact, when the interpolation is made from *any* $2n + 1$ values of x, as $x_1, x_2, \ldots$, situated *symmetrically about* $h/2$, the error in approximating to a polynomial of degree $2n + 1$ is of the form $C(x - x_1)(x - x_2) \ldots$, which is an odd function of $x - h/2$, so that its integral between 0 and h vanishes.

[1] For further illustration of the importance of the fixed origin in the problem as proposed, the reader may contrast it with the case of the equation

$$\int_{x=-h/2}^{x=h/2} f(x)dx = (h/6)[f(-h/2) + 4f(0) + f(h/2)]$$

for which the most general continuous solution is $f(x) = a + bx^2 +$ (an odd continuous function).

be bounded by a three-cusped hypocycloid; if we consider only convex areas, perhaps the result will be an equilateral triangle. I have no indication of a proof.

2. For every closed convex curve of area P there is an n-sided circumscribed polygon of least area Q and an inscribed polygon of greatest area R. For a fixed value of the integer n and for all convex curves, what is the upper limit of Q/P and what is the lower limit of R/P? I have succeeded only in proving that for the case $n = 3$ the upper limit of Q/P is 2.

3. Let the area of a simple closed curve A be a. Remove from A the greatest possible area a_1 similar to another simple closed curve B. From the remaining figure remove the greatest possible area a_2 similar to B. Continue this process indefinitely. Is it or is it not true that

$$a_1 + a_2 + a_3 + \ldots = a?$$

I have proved the statement to be true in the special case that A is convex and B is a circle.

4. Let a given closed convex curve K have the property that a given triangle whose angles are incommensurable with π can be revolved completely within K (see part 1 of this question), always remaining inscribed to K. What may the curve K be? Can any other curve except a circle satisfy the conditions?

The questions contained in no. 39 were proposed by Professor S. Kakeya, and were contributed to this department at the request of the editors. The questions have aroused considerable interest, but up to the present have not been solved.[1]

41 [*1920*, 365; *1921*, 126]. A reader asks for an elementary proof of the following two propositions in number theory, either of which can readily be obtained from the other:

Every positive integer of the form $8n + 3$ *is the sum of three odd squares.*

Every positive integer is the sum of not more than three triangular numbers.

According to Bachmann[2] these results have only been proved by means of the theory of ternary forms.

42 [*1921*, 65, 126]. In connection with the questions of Kakeya, Professor W. B. Ford is led to the following inquiry: A line-segment AB is to be moved in its plane to a new position $A'B'$. How should this be done in order that the area generated may, to the greatest extent possible, be passed over three times?

Professor Ford has proved that, if the generated area is to be passed over, to the greatest possible extent, but *two* times, AB should be moved so that its instantaneous center of rotation is always on its right bisector.[3]

43 [*1921*, 260]. Is any rapid method known for the evaluation of the Sylvester determinant met with so often in elimination by the dialytic method? It would seem that there must be, both on account of its interesting shape, and of its frequent occurrence.

45 [*1921*, 305]. Is every non-trivial solution in integers of the equation $t^3 = x^3 + y^3 + 1$ expressible in the form $x = 9r^4 - 3r$, $y = 9r^3 - 1$, $t = 9r^4$? If there are non-trivial solutions not expressible in this form, can a general solution be found?

This problem arose out of a note by H. C. Bradley (*1921*, 307) in which he obtained the solution quoted by specialization of the formula

$$x = -(a - 3b)(a^2 + 3b^2) + 1, \qquad y = (a + 3b)(a^2 + 3b^2) - 1,$$
$$u = -(a^2 + 3b^2)^2 + (a + 3b), \qquad v = (a^2 + 3b^2)^2 - (a - 3b),$$

which was shown by Euler and Binet to be (except for a common multiplier) the most general rational solution of the equation $x^3 + y^3 = u^3 + v^3$, a and b being any rational numbers.[4] This leads at once to the general *rational* solution of the

[1] See, however, Question **42**, reprinted in this number.

[2] *Niedere Zahlentheorie*, Leipzig and Berlin, 1910, Teil 2, p. 325. See also Dickson, *History*, vol. 2, Chapter 7.

[3] W. B. Ford, "On Kakeya's minimum area problem," *Bulletin of the Amer. Math. Soc.*, vol. 28., 1922, pp. 45-53. Other references are given in this paper.

The editorial note in this MONTHLY (*1921*, 126) quotes Professor Ford's conclusions inaccurately.

[4] Carmichael, *Diophantine Analysis*, p. 65; Dickson, *History*, vol. 2, p. 555.

more special equation, so that the problem awaiting solution is one of *integral* values only.

DISCUSSIONS.

The following note is a reply to that of Professor E. T. Bell (*1920*, 413), in which the use of mathematical induction in elementary teaching was criticized on logical grounds. Professor Bell's position was that the method required the postulation of a special logical principle, and was therefore unsuited to beginners.

On Proofs by Mathematical Induction.

By R. S. Hoar, South Milwaukee, Wis.

The objections to the usual method of stating mathematical induction, raised by Professor Bell in the Monthly of November, 1920, seem to call for some further discussion.

When I was a boy, the part that confused *me* was: "We first establish a theorem for $n = 1$; then we show that, if it is true for $n - 1$, it is true for n." But, if n is 1, $n - 1$ must be zero. We have not proved it for zero, and we already know that it is true for n. Of course, such an attitude is absurd, even though natural. But is it absurd to object that we never prove the proposition for $n - 1$ equals anything (but merely for n equals something), and therefore never lay the foundation for the second step?

Why not state the process as follows? "We first show that, if the proposition is true for n, it is true for $n + 1$; then we show that it is true for a particular value of n, namely $n = 1$; then, since any whole number can be reached from any preceding whole number by successively adding 1 a finite number of times, the proposition is true for n equals any whole number."

The false proof that any assemblage contains an infinity of members, cited by Prof. Bell, is not based on mathematical induction in the form objected to by him, but rather is based upon the layman's substitute for mathematical induction; namely, that if a proposition is true for $n = 1$, $n = 2$, $n = 3$, etc., repeated a "reasonable" number of times, then it must be true for any value of n.

The fact that such "proofs" are current among freshmen is probably due to the failure of their teachers to emphasize the necessity of taking the abstract general step from n to $n + 1$; just as the currency of "proofs" that two equals one, is due to the failure of teachers to introduce non-divisibility by zero at the very start, as a fundamental part of the concept of division, instead of dragging it in later as a rather lame exception, in order to defend division from the onslaughts of two-equals-one.

A real example of the non-applicability of mathematical induction would be the series set forth on page 15 of Huntington's *The Continuum* (Cambridge, Mass., 1917):

$$1_1\ 2_1\ 3_1 \cdots; \qquad 1_2\ 2_2\ 3_2 \cdots; \qquad \cdots.$$

Perhaps it would be well for teachers to exhibit both this series and the "infinite assemblage," in order that the students may realize the necessity for all three steps in mathematical induction.

REMARKS BY THE EDITOR.

Mathematical induction may be criticized with regard either to its logical basis or to its practical intelligibility. Professor Bell's contribution was of the former kind, and Mr. Hoar's of the latter. Philosophy apart, the difficulty about a proof by induction will be in making "the abstract general step from n to $n+1$." Certainly the irregular use of n and $n-1$ to which allusion has been made is a needless source of confusion, though perhaps not very serious in the long run. There are of course many to whom the very idea of the abstract general step is a hidden mystery. Such persons constitute a perennial problem in elementary teaching.

While we have the subject before us, it may be of interest to note that mathematical induction is only one of a number of processes by which theorems are extended over a wider range according to the following general plan:

It is known of a theorem T: (1) that T is true for numbers lying in a given range N_0; (2) that whenever T is true in any range N, T is also true in the range $\Omega(N)$ derived from N by means of the operations forming the set Ω. Then T is true for the numbers of any range which can be reached from N_0 by a finite number of successive applications of Ω.

Special cases are:

(a) N_0 consists of the number 0, and Ω consists of the operation which, applied to any set of consecutive integers, gives the same set with the next higher integer added. This is ordinary mathematical induction, proving results for all positive integers.

(b) The same except that it is the next lower integer that is added. This proves theorems for all negative integers.

(c) N_0 consists of all integers; Ω includes division by any positive integer. This proves theorems for all rational numbers.

(d) Cases where Ω includes the operation of adjoining to a set of numbers the limits of the set.

Some of the proofs given for the binomial and exponential theorems for general real exponents are combinations of these four types of argument.

(e) N_0 is the interval $(0, x_0)$; Ω consists of the operation which replaces any interval $(0, x)$ by $(0, f(x))$, where f is a positive function. This extends theorems to the interval $(0, u)$, where u is the least upper bound of $(x_0, fx_0, ffx_0, \cdots)$.

The special case of (e) in which $f(x) = x + k$, leading to theorems on all positive numbers, was given in detail by Y. R. Chao,[1] and illustrated with a "thin end of the wedge" fallacy. In the illustration x_0 is the measure of a dose of an intoxicant, too small to be harmful. By using a plausibly small constant k it is argued that any amount is harmless. This optimistic conclusion is reached by correctly applying the method to rather questionable assumptions.

Errors perhaps occur more often from incorrect use of (e) with data which are reasonable enough. For an extreme example: An athlete is known to be

[1] "A note on 'Continuous Mathematical Induction,'" *Bulletin of Amer. Math. Soc.*, vol. 26, 1919, p. 17.

able to jump any height up to five feet. However high he may jump, he could always have gone a little higher (which is at least as easy to admit as the contrary). Hence he could jump any height. But in reality, $x_0 = 5$ and $(5, f(5), ff(5), \cdots)$ is a *convergent* sequence whose limit is probably less than 7.

RECENT PUBLICATIONS.

REVIEWS.

Dynamic Symmetry, the Greek Vase. By JAY HAMBIDGE. New Haven, Yale University Press, 1920. 161 pages, small folio. Price $6.00.

A mild sensation has been caused among classicists, artists and the philosophers of æsthetics by the doctrine of "dynamic symmetry" as recently enunciated and promulgated by Mr. Jay Hambidge and his converts. Certain ambitious claims have been made which have been greeted, possibly with scepticism and repugnance by some artists and classicists, but certainly with interest and sympathy by many readers, including the Yale School of Fine Arts, and the members of various museum staffs. The exposition has been clothed in a certain amount of numerical terminology, so that many who would, perhaps, desire to approach the subject with an open mind may have been discouraged at the start and led to regard the matter as too technical for the comprehension of any one not deeply versed in the intricacies of higher mathematics.

So far as the reviewer is aware, this work has not received from professional mathematicians the attention that it deserves, particularly since any adequate criticism of the results claimed can come only from one prepared properly to appraise the mathematical background of the argument. The book under review is assuredly not intended as a reference work in geometrical analysis, but in its wider philosophical aspects, its claim for the historical and logical application of geometry to design must merit the consideration of any one who is interested in the place of mathematical discipline among the cultural developments of civilization.

Let us first cite some passages in which Mr. Hambidge enunciates his claims. "Some twenty years ago, the writer, being impressed by the incoherence of modern design and convinced that there must exist in nature some correlating principle which could give artists a control of areas, undertook a comparative study of the bases of all design, both in nature and in art. This labor resulted in the determination of two types of symmetry or proportion, one of which possessed qualities of activity, the other of passivity. For convenience, the active type was termed dynamic symmetry, the other static symmetry. It was found that the passive was the type that was employed most naturally by artists, either consciously or unconsciously; in fact no design which could be recognized as such—unless indeed it were dynamic—would be possible without the use, in some degree, of this passive or static type. . . . A study of the basis of design in art shows that this active symmetry was known to but two peoples, the Egyp-

tians and the Greeks, the latter only having developed its full possibilities for purposes of art. The writer believes that he has now recovered, through study of natural form and shapes in Greek and Egyptian art, this principle for the proportioning of areas. . . . At some time during the Sixth or Seventh Century B. C. the Greeks obtained from Egypt knowledge of this manner of correlating elements of design. In their hands it was highly perfected as a practical geometry, and for about three hundred years it provided the basic principle of design for what the writer considers as the finest art of the Classic period. Euclidean geometry gives us the Greek development of the idea in pure mathematics; but the secret of its artistic application completely disappeared. Its recovery has given us dynamic symmetry—a method of establishing the relationship of areas in design composition. . . . This idea (the application of areas) is quite unknown to modern art but that it is of the utmost importance will be shown in this book by the analysis of Greek vases. . . . Analysis of Greek and Egyptian composition shows that the artist always worked within predetermined areas. The enclosing rectangle was considered the factor which controlled and determined the units of the form. . . . Analysis of any fine Greek design is sure to disclose an arrangement of area which produces the quality of inevitableness, so conspicuously absent in modern art."

Mr. Hambidge when speaking of mathematical notions does not employ in all particulars customary mathematical language. This may be due to a desire to escape from the unnecessarily technical, but in any case it makes the reading more difficult for the mathematician and probably meaningless for the average artist. When he speaks of "proportioning areas," it has no reference to the familiar artistic principle often discussed under the term, space balance, or the balance of areas, which will be referred to later. The "proportioning of areas" means here, in effect, the determination of significant points and horizontal lines within a rectangular diagram by use of geometrical methods.

A condensed summary of the claims made for dynamic symmetry as applied to the Greek vase may be expressed in the following two statements concerning the profile curve of the lateral elevation of the vase, including the handles if such there be. On the negative side—except for the fundamental bilateral symmetry of the design, equipartition of ordinates and of abscissas is avoided, and any regularity of linear division that may exist avoids simple commensurate ratios. On the positive side—while commensurate ratios are avoided, the incommensurate ratios occurring are such that their squares or rational combinations among these squares are simply commensurable. This means incidentally, as every mathematician knows, that the significant points and lines can be located by an elementary use of ruler and compasses.

To what extent has Mr. Hambidge established his theory? As in the case of other enthusiasts there is among his remarks no hint of doubt or failure. He is convinced whole-heartedly of his claims and expects the reader to acknowledge as ample the evidence displayed. Analysis is given in drawings or tables of many examples and types. A count of these scattered throughout the text

shows the following distinct specimens, here collected and listed by type: amphora (15), small cup (1), dinos and stand (1), hydria (3), kalpis (7), kantharos (2), bell krater (3), volute krater (1), kylix (19), lekythos (10), olpe (3), oinochoe (2), pelike (3), perfume vase (1), psykter (1), pyxis (3), skyphos (14), stamnos (2), a total of 91 cases analyzed. While this is of course an inconsiderable proportion of the extant Greek vases, yet results found to hold uniformly in the examination of the first 91 random samples may be fairly assumed to establish the presumption that the thesis is valid throughout.

In emphasizing the beauties of his new theory, Mr. Hambidge appears to have overstressed the lack of principle in current theories of design. The inevitableness and geometrical background that he claims as the characteristic features of the doctrine of dynamic symmetry have not been rediscovered and expounded here for the first time. Let me quote from E. A. Batchelder's conservative *Principles of Design:* "A finished and satisfactory design should never suggest the question, 'Can I remove this feature'? or 'Is that line necessary'? There should be no question about the importance of each line involved" (page 25). Again, "The law of balance in so far as we have gone may be concisely stated as follows: 'Equal measures of equal contrasts will balance at a point midway on a line connecting the centers of the measures. Unequal measures of equal contrasts will balance on a line connecting the centers of the measures at distances which will be in inverse ratio to the measures'" (page 42). "The Indian has not yet acquired sufficient grasp on the principles of his art to allow his imagination to express itself unrestricted by the limitations of geometry. Japanese art, on the other hand, to the casual observer, contains very little of geometry; but the student of the art of Japan knows that the work of that country rests just as firmly upon the fundamental principles of design as the work of any other nation or period" (page 44).

Mr. Hambidge seeks to forestall certain obvious questions. Most of his measurements are made by the members of the staffs of the museums which contain the vases analyzed, and he takes care to remark in some cases that the drawings were not made to fit the measurements expected, but were first executed carefully to scale, and subsequently analyzed. A not unnatural basis for scepticism he disposes of as follows: "The shape of the vessel is determined with great care" and "shows unmistakably that the picture (the red-figured decoration upon the vase) is secondary. . . . The idea that so much care should be taken to proportion such a commonplace article as a clay pot will probably strike the average reader as fanciful. And it would be so if ordinary pottery were under consideration. The vases considered here, however, are Greek, and the Greek vase is unique. . . . Volumes have been written about the pictures found on Greek pottery, but the shape or form of the vase, which is of much greater importance, has been almost entirely neglected. . . . To the ancient Greek, the form of the vase was of vital importance, the vase painting was usually of secondary importance, a fact made clear by the great preponderance of signatures of potters over those of painters."

Mr. Hambidge has assuredly succeeded in exhibiting certain vase forms in which the simple mathematical proportions of the parts are too convincing to be dismissed as accidental. In particular his single example of a small cup shows each of its important points identified by the simplest sort of construction. The fact that many of the vases are "exactly" inscribed in a square, that so often among the examples discussed the width of the foot is "identical" with that of the lip seems hardly likely to be a mere coincidence. However the reviewer at least is far from being convinced as to the general claim.

The question reduces as to whether evidence has been shown that Greek vases were regularly designed with mathematical accuracy in all essential parts to accord with measurements in a diagram constructible with ruler and compasses. The independent incommensurables referred to by Mr. Hambidge are, in particular, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, the so-called "whirling square" ratio being the positive root of the equation, $1/(x - 1) = x$, namely, $\frac{1}{2}(1 + \sqrt{5})$.

I shall first point out what I regard as fatal flaws in the argument for the historical significance of dynamic symmetry. To put the matter briefly, it seems that the reader is at liberty to question both the exactness and the adequacy of the analyses submitted in the book.

The impression is conveyed that most measurements displayed check to a remarkable degree of exactitude with the drawings exhibited. This impression is heightened not only by explicit statements to that effect but also by the careful pointing out and explanation of a few noticeable but not serious departures. Occasional remarks however have slipped into the text that invite a contemplative pause. Thus, "The percentage of error is much smaller in the bronzes than in the pottery." The term "error" is clearly used to denote discrepancy between assumed analysis and measured dimensions. It is enlightening that no hint of recurrent percentage error in the pottery is made until it is desired to emphasize the accord of the bronzes with assumed canons. A detailed explanation is offered as to why the handles of the kylikes should show such random heights in their relations to the bowls. The explanation sounds more ingenious than plausible, but in any case admits an irregularity as to heights that is thereafter ignored. One wonders why out of a list of fifty-four analyzed kylikes, a table of but seventeen is given. One suspects, perhaps unfairly, that the remaining examples fail to bear out the theory in so obvious a manner. Throughout the treatise the reviewer feels that there is a seeming inclination to overlook or condone apparent discrepancies, and little suggestion of an impartial statistical attitude that would invite and compel conviction.

The adequacy of the analysis may be criticized on many counts. The vase profiles are in general interesting curves not identified by means of a few points and tangents. One is led at once to inquire whether the analysis covers the determination of these most important features of the design. The author says, "No mathematical curves have, so far, been found in Greek art. . . . Curves were apparently drawn by tangents in this manner (with assigned tangents at certain points) all through the Greek classical period." Certain portions in a

design which do not lend themselves to convenient analysis are referred to as "loosely rendered." A cursory survey of the analyses submitted is pleasantly satisfying, but a closer inspection frequently reveals what seems little more than random selection. Certain conspicuous points in the design are located in the system, but others quite as important are not included among those present and accounted for. It is particularly striking that of two vases practically identical to the eye, the analysis should in one case include certain points of the vase profile, while for the other a set notably distinct should be chosen for special mention. One feels instinctively that the featured points were signalized *a posteriori* as being those which appeared to fit passably into an inflexible and inadequate mould. In several cases the drawings exhibit the actual asymmetry of the material vase, and lines determining intersections for the right half of the vase would fail for the left, leaving the actual significance of the points plotted entirely in doubt.

But there are much stronger reasons for challenging the principal argument of the book. One may be prepared to agree with Mr. Hambidge in believing that Greek architecture and architectural design employed dimensions accurately laid out by ruler and compasses in incommensurable as well as in commensurable ratios. An undertaking in which the designer and the artisan are distinct, such as building a temple, gave ample scope for the use of working drawings, laid out carefully by such geometrical drafting instruments as were available. Temples were not essentially commercial and were not multiplied daily. There is however ample evidence that the Greek vases, except only for some of very large dimensions, were turned on a potter's wheel; that the designer was in many of the finest examples also the executor, and could therefore judge with his eye as to esthetic excellence as easily as measure the wet clay against a scaled drawing. That no two Greek vases are known of identical form, despite the rigid commercial types into which they fell, is almost proof positive that no exact measurements were intended. Vases were manufactured by the hundreds and thousands and for strictly utilitarian ends. Their beauty and their charm were merely reflections of the maker's pride in careful and honest workmanship. All authorities agree that the finest Greek vases were never intended as mere decorations. Their design as well as their execution was in the hands of the metics, a skilled but usually poorly educated class of artisans. It is next to inconceivable to me that a complicated working drawing was made afresh for each vase, that differences scarcely noticeable to the eye were purposely secured at great labor by extensive and thoroughgoing alterations in the pattern of construction lines. Yet this is what Mr. Hambidge would have us believe. Granted that the curves themselves were not constructed by mathematical methods, what right have we to suppose that the critical points of the design need have been so chosen?

It remains to dispose of the immediately pertinent inquiry as to how one can account for the remarkable analyses submitted by Mr. Hambidge. My answer is a simple one, perhaps too simple to appeal to the fancies of any who have caught that distinguished illustrator's enthusiasm for dynamic symmetry. The

author uses the square roots of 2, 3, and 5, and all that can be obtained from these as linear expressions with rational coefficients. He reserves also the privilege of utilizing other quadratic surds in the same manner but appears to find these sufficient. Among the infinite set of numbers between zero and unity that can be so obtained, I have noted 53 distinct ones which the author mentions explicitly in the course of the text. Two of these, namely, $2\sqrt{5} - 4$, which he writes as .472, and $\sqrt{2/3}$ which he writes as .4714 (the text employs decimal notation throughout), differ by approximately one part in fourteen hundred. Thus by using this simple difference alone we can obtain something like fourteen hundred intervals between zero and unity. The reply to the query is then this obvious one, namely, that Mr. Hambidge has provided himself with so bountiful a list of incommensurables that any observed measurement may be expressed by the aid of these to a degree of precision beyond the most fastidious ambitions of the potter. There is a single table in the book exhibiting certain relative dimensions of several vases. Without being here concerned with the particular significance of the data selected, we shall write down for comparison two possible sets. The first row is selected from corresponding entries in the table which are supposed to illustrate, by their numerical form, "root two figures." These are to be contrasted with those of the second row which has been selected as illustrating figures which are not "root two" in theme. The numbers have been chosen from the columns given by the author, although it is but fair to remark that the numbers here given do not all happen to occur in a common row in the book. These rows are as follows:

	3.4142,	2.4714,	.9393,	.2929,	.7071,	.3535;
and						
	3.382,	2.472,	.927,	.309,	.691,	.354.

These dimensions would be regarded by the author as characteristic of vases of markedly different type. It is beyond the present credibility of the reviewer to conceive as impossible that the analyst in measuring two vases, one with the ratios of the first row, and the other with the ratios of the second row, might not utilize his assumption of percentage errors and fall into the "confusion" of mistaking these two shapes as negligible variants from a common design. Mr. Hambidge even remarks in one connection that the analysis may not be unique, although perhaps he there refers to the geometrical steps followed and not to the numerical data assigned.

Unlike most books on design, this text explicitly warns the inquiring artist against applying purely esthetic criteria, informing him that to emulate Greek perfection, it is essential and apparently sufficient to acquire facility in manipulating the elementary arithmetical incommensurables. The author does not hesitate to praise one vase above another, both of which are supposed to comply equally with the canons of numerical exactitude. How to design a *good* vase as distinguished from an exact but ill-conceived one is not discussed. And is not this the most important question suggested by the whole discussion?

For its probable future influence, its suggestive analyses, its rich lore of illuminating and interesting mathematical and archeological references, and not least for its exquisite half-tones of rarely charming vases, this book is recommended for the attention and study but not the blind acceptance of such as appreciate science and art.

ALBERT A. BENNETT.

Calculus and Graphs—Simplified for a First Brief Course. By L. M. PASSANO. New York, The Macmillan Company, 1921. 12mo. 8+167 pp. Price $1.75.

As every one is aware, who has had the pleasure of teaching the subject of calculus, many features of the subject present no essential difficulties even to students who have shown no special aptitude along mathematical lines. The mere technique of differentiation can be readily acquired by students who cannot learn facility in the manipulation of trigonometric identities. The chief difficulty in solving elementary problems seems to lie in the lack of ability displayed by the student in converting a verbally formulated problem into tractable equations. This is a difficulty that is earlier presented by the subject of algebra, and has nothing to do with any one particular branch of mathematics. As long as verbal problems are offered for analysis, there will be students who find the subject difficult. In the present book there has been no effort to spare the student in this direction. Since the development of clear reasoning is one of the primary purposes in any mathematical course, this book assuredly fulfils its principal aim.

Of the other difficulties which the subject sometimes presents may be mentioned the extensiveness of the field of applications, and questions of rigor connected with the notion of limit. In the text under discussion, the material is wisely restricted so as to make possible an emphasis upon essentials without which the student can receive no benefit from the course. The author gives us the following warning in his preface: "The aim has been to make the student understand the subject; not to write a book that would satisfy meticulous mathematical pedantry. In so doing the author may have in places sacrificed logical detail to simplicity of presentation, but never, he hopes, accuracy of statement. In the opinion of the writer a too rigidly logical proof with its paraphernalia of subscripted Greek letters is out of place in an elementary first course in calculus, for the reason that the student never understands such a proof. Or if by arduous effort he does grasp its meaning, it is at the expense—in time and labor—of other things that are more important and far more useful." In view of this announcement the reader need not be surprised to see infinite series used where necessary without a word about convergence; but some other features are not so readily explained. On page 15, we read, "We do not know whether $x + 2$ is plus or minus zero; that is, whether $x + 2$ is an exceedingly small positive or an exceedingly small negative number." On page 19, and repeatedly throughout the book, "The increment of the function can be made smaller than any number we may assign, however small, by making the increment of the independent variable small enough . . .," where the word "number" is apparently expected to mean

"positive number." On page 80, the phrase "any number of parts" is to be understood as meaning "any number of equal parts." The book is almost free of definitions. A search through the text has failed to reveal any definition or even hint as to what may be meant by "element" until on page 93, we read, "To make use of this new tool [the definite integral] all that is necessary is to form the element of the thing to be summed; to get a differential expression representing, in terms of some independent variable, any one of the terms to be summed. This element being found and formed, the remainder of the process consists in integrating, and substituting the limits as shown in (40)," while on page 122, we read "of another little element." The problems considered on maxima and minima are of the usual sort, and, as is obvious, examples may readily be constructed in which an extremal is obtained at the end of the interval of definition. That no such examples should happen to be incorporated in the particular collection here given does not justify the statement: "For the function to reach a maximum or a minimum value, must $dy/dx = 0$. This, as we say, is a *necessary* condition for a maximum or minimum value of the function." One need hardly state that the notion of an interval of definition is not considered. The student is asked to *prove* that a locus is a circle, by plotting its points (p. 45, Ex. 26).

A more practical criticism, from the viewpoint of the text, is one of little logical import and refers to the form of the discussion. Answers obtained in the illustrative examples are not so labeled. The discussion consists in most cases of a mere sequence of equations with no hint as to what is assumed and what derived and whence. Many of the discussions continue uninterruptedly for too many pages for any normally impatient student to desire to understand them. This is particularly pertinent to section 20. What student will care to follow the advice: "Formula (63) can best be learned in words. Thus: The integral of the product of two factors, one of which can be integrated separately, *equals* the product of the integral of that one factor times the other; *minus* a new integral consisting of the same integral of the one factor times the derivative of the other"? The use made of units in a book intended for engineers, or rather the abuse of them, is a surprise—for example, "200 π radians minute," without even the apology of an abbreviation, stands for a velocity in radians per minute, and similarly throughout the book. The misprints are few. Only one bad case was noted, where a letter of the wrong font was used on page 124, 5th line. Something like half of the formulas appear with no terminal punctuation, so that this unusual procedure appears to be intentional, although not observed consistently.

Despite all of these incidental objections and many more that are trivial, the general arrangement of the material and the choice of exercises are such as to suggest that the book might prove a very satisfactory text in the increasing number of institutions that are giving first year students a taste of one of the richest treats afforded by mathematics, the calculus.

ALBERT A. BENNETT.

NOTES.

A comprehensive catalogue of books on arithmetic published before 1501 A. D. is in active preparation by K. A. PEDDIE, who has published other scholarly bibliographical works. This catalogue will supplement D. E. SMITH'S well-known *Rara Arithmetica* (Ginn & Company) and DE MORGAN'S *Arithmetical Books from the Invention of Printing.* Subscriptions at 1 £ 1 s. are desired to insure publication; address Grafton and Co., Coptic House 7 and 8, London W. C. 1.

In *Science*, September 30, 1921, attention is called to a bibliography of the theory of relativity that is being compiled by F. E. BRASCH, now chief librarian at the J. J. Hill Reference Library, St. Paul, Minn. The aim is to make the bibliography as extensive and complete as possible, each entry to include not only author, title, source, and date, but also a short abstract, note, or review indicating the principal idea the author has conveyed. It would seem desirable to have this bibliography published, but at the time of the notice in *Science* no provision had been made for this.

Father J. G. HAGEN, the gracious director of the Observatory of the Vatican, has almost ready the manuscript for the fourth volume of his *Synopsis der höheren Mathematik* of which volumes 1–3 were published in Berlin (Dames), 1891–1905. The older members of the American Mathematical Society will recall with pleasure Father Hagen's presence at their meetings.

Four of the seven volumes of PAUL TANNERY'S *Mémoires Scientifiques* have already appeared (Toulouse, E. Privat), the fourth volume, *Sciences exactes chez les Byzantins*, in 1920. Professors ZEUTHEN and HEIBERG have been the editors. Since Zeuthen's death Professor GINO LORIA has acceded to Madame Tannery's request to take his place as an editor.

Professor LORIA is preparing the manuscript for a third volume of his work on algebraic curves, *Spezielle algebraische und tranzendente ebene Kurven, Theorie und Geschichte* (Teubner, Berlin, 1910–1911). It is to deal with gauche curves.

We have already made note of several institutions which have collected mathematical papers, published by members of their faculty (*1921*, 177). In 1920 was published the first volume of *Publications de L'Institut de Mathématiques de l'Université de Strasbourg.* It contained four memoirs by M. FRÉCHET, H. VILLAT, A. VÉRONNET, and G. VALIRON. The first of these memoirs was originally published in *Transactions of the American Mathematical Society.* The second volume of the *Publications* was for 1921, and was published in May, 1922. Only 25 copies each of volumes 1 and 2 were published.

With the first number of the *Journal de Mathématiques pures et appliquées* (Liouville) for 1922 a new series (the ninth) commences under the direction of Jordan's successor, H. VILLAT, professor at the University of Strasbourg. The number contains a portrait of Jordan and an obituary notice by Villat (see *1922*, 139), who, it will be remembered, was the editor of the large volume of the proceedings of the Strasbourg Mathematical Congress. Villat is also to be director of the *Nouvelles Annales de Mathématiques* which is to start again in

October, 1922. The second number of the *Journal*, published in May, is about double the size of the first number and is entirely devoted to papers on relativity.

That interest in the applications of mathematics to commercial and professional work has been further stimulated in France by the war is evidenced by the number of recent books dealing with such subjects. For example, one lesser publisher (Dunod, Quai des Grands-Augustins 47 et 49, Paris) alone lists, among others, recent books with the following titles: *La culture générale des jeunes gens se destinant à l'industrie; Les mathématiques de l'élève ingénieur*, Part I (1920); *Effets gyroscopiques et méthodes vectorielles* (1920); *Les mathématiques de l'ouvrier moderne*, 2 volumes; *Trigonometrie rectiligne* (1920); *Dessin industriel. Cours pratique de projections* (1919); *Traité de dessin géométrique* (1919); *Cours de comptabilité* (1920); *Notions sommaires de comptabilité industrielle* (1920), *Cours élémentaire de méchanique industrielle* (three volumes, 1919–1920); *Comptabilité commerciale* (two volumes, 1918–1920); *Les calculs commerciaux et financiers* (in press). Few of these would be of practical service in this country and for this reason no mention of author, paging, or prices need be made.

ARTICLES IN CURRENT PERIODICALS.

ANNALI DELLE UNIVERSITÀ TOSCANE, new series, volume 4, fasc. 2, 1919: "Sulle curve piane algebriche che ammettono una curva data come prima polare" by G. Albanese, 33–41—Volume 5, fasc. 8, 1920: "Sulla moltiplicazione delle serie" by U. Dini, 253–274.

ISIS, volume 4, no. 1, 1921: "Different types of mathematical history" by G. A. Miller, 5–12; "Note on the Fahrenheit scale" by F. Cajori, 17–22; "Introduction to the history and philosophy of science (preliminary note)" by G. Sarton, 23–31; "The principle of symmetry and its applications to science and to art" by G. Sarton, 32–38 [Discussion of ideas of F. M. Jaeger, D'A. W. Thompson, T. A. Cook, J. Hambidge, D. W. Ross]; "*Encyclopédie des Sciences mathématiques pures et appliquées*" by G. Sarton, 39–40 [Protest against the slowness of this publication]; Reviews of: J. Ruska, "Zur ältesten arabischen Algebra und Rechenkunst" (*Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Philosophisch-historische Klasse*, pp. 1–125, 1917) by L. C. Karpinski, 67–70; P. Boutroux, *Les Principes de l'Analyse Mathématique*, tome second (Paris, 1919) by J. M. Child, 96–107; L. E. Dickson, *History of the Theory of Numbers*, volume 2, by J. M. Child, 107–108; H. Wieleitner, *Geschichte der Mathematik* (II. Teil, II. Hälfte) by D. E. Smith, 109.

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 13, October, 1921: "Some pitfalls in the theory of groups" by G. A. Miller, 161–165—December: "On a certain theorem in geometry" by F. H. V. Gulasekharam, 210–218; "Mathematics in India, then and now" by V. Sankaran, 219–224 (to be continued) ["It is a characteristic of the present times in India that we feel as if we have been shut out from all knowledge of the glories of this ancient land in the palmy days of yore. The cry is now 'Look back into your past history and you will learn more than what you can by merely studying what you are taught in schools.' In every department of knowledge this looking back into the past is necessary. We have begun to realize how advanced our fore-fathers were in the study of the Humanities, Political and Social Sciences, Arts and Philosophy. It is therefore, but natural to expect that in other departments of knowledge also, they should have reached a tolerably high standard of advancement and efficiency."]; "Interference method in astronomy" by T. P. Bhaskara Sastri, 230–231.

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES, 8th series, volume 4, fasc. 4, 1921: Statement by C. Jordan (who has since died), on retiring from the active management of this Journal after 35 years of service; announcement of memoirs to be published in the near future, insert 1–4; "Integral solutions of the equation $\xi^2 + \eta^2 = \zeta^2$ in the quadratic realms of rationality" by H. Hancock, 327–341—9th series, volume 1, fasc. 1, 1922: "Camille Jordan," insert I–IV (see *1922*, 138).

MATHEMATICS TEACHER, volume 15, March, 1922: "The psychology of the equation" by E. L. Thorndike, 127–136; "Reaction vs. radicalism in the teaching of mathematics" by G. W.

Myers, 137–146; "The definition of similarity" by G. W. Evans, 147–151; "The place of elementary calculus in senior high school mathematics" by N. B. Rosenberger, 152–156; "Experimental geometry" by G. A. Harper, 157–163; "How can I bring the soul of mathematics to my pupils" by A. H. Huntington, 164–171; "Papers by pupils of the plane geometry classes of Fullerton Union High School" by Lena E. Reynold, 172–181; "Discussion" by L. F. Babcock and P. Huffman, Jr., 182–184; New books, 185–188; News and Notes, 189–190.

NATURE, volume 109, February 23, 1922: "The pioneer of non-euclidean geometry," 232–233 [Review of G. B. Halsted's *Girolamo Saccheri's "Euclides Vindicatus."*]—March 2: "The directive tendency of elongated bodies" by W. D. Lambert, 271–272; "Mathematics in Japan" by G. B. M(athews), 287–288—March 16: "Greek mathematics" by D'A. W. Thompson, 330–334 [Review of Heath's *History of Greek Mathematics*, Oxford, 1921. "Were this book only for the mathematician it would be no book for me; but it is a great deal more. It is for all who care for the historical aspect of science; it is for all lovers of Greek, for mathematics is a true 'Legacy of Greece,' and is interwoven through and through with Greek thought and philosophy. . . . There is a vast deal of information in Heath's book, clearly set forth and orderly arranged; we have nothing to compare with it in English, and Gino Loria's 'Scienze esatte' is its only serious rival abroad. I am inclined to think that Loria paints history with a broader brush, while Heath excels in his account of individual mathematicians; but I cannot help thinking that Heath, who has attained such complete and acknowledged success in his editions of Euclid and the rest, must have found that in this history he had struck a harder task than any he had tried before. We may know more of the history of mathematics than of any other science, but the lacunæ are immense, and tradition is poor material for the historian. Moreover the historical aspect is somewhat uncongenial to the mathematician, if only because (as Eva Sachs says) history deals with *das Werden*, and mathematics with *das Sein*! When Sir Thomas Heath deals with Euclid, Apollonius, Archimedes, Diophantus, Hero or Pappus, he gives us in a few pages all we could expect by way of epitome of the trend, the method and the results of their labours. But his book pursues its steady, instructive course with little digression, allusion or anecdote, and with curiously little bibliographical information such as he puts abundantly into his other books. Surely one of the objects of a text-book is to guide the student to what it does not and cannot contain! Some of us, I think, would have liked a little more digression or even gossip. . . . The simple fact is that Sir Thomas Heath has given us so much, and it is all so good, that he makes us ask for more."] —March 23: "The theory of relativity in relation to scientific method" by Dorothy Wrinch, 381–382 [Paper read before the Congress of Philosophy in Paris on December 29, 1921]—April 8: "Mathematical analysis" by G. H. Hardy, 435–438 [Review of Hobson's *Functions of a Real Variable*, volume 1, 2nd ed., (Cambridge, 1921); Carslaw's *Introduction to the Theory of Fourier's Series*, volume 1 (London, 1921); Edward's *Integral Calculus*, volume 1 (London, 1921)]; "Dr. G. B. Mathews" by W. E. H. B., 450–451—April 15: "The fourth dimension" by S. Brodetsky, 474–475 [Review of *The Fourth Dimension Simply Explained.* A Collection of Essays selected from those submitted in *The Scientific American's* Competition (New York, 1913, London, 1921)]; "Pythagoras's theorem as a repeating pattern" by P. A. MacMahon, 479—April 22: "Mathematics and public opinion" by G. B. M(athews), 520–521.

LA NATURE, volume 50$_1$, February 11, 1922: "L'Univers stellaire" by É. Belot, 87–91—March 18: "Le calendrier perpétuel vivant" by E. Singer, supplement 86–87.

OBSERVATORY, volume 45, March, 1922: "Meeting of the Royal Astronomical Society, February 10, 1922," 65–69 [Address of the President, A. S. Eddington, in presenting the gold medal of the society to J. H. Jeans]; "An eclipse of the moon as seen from the moon" by L. Richardson, 69–71 [Paper read at the meeting of the British Astronomical Association].

PERIODICO DI MATEMATICHE, series 4, volume 1, November, 1921: "Che cosa contiene la Géométrie di Cartesio?" by E. Bompiani, 313–325; "La prospettiva e lo sviluppo dell'idea dei punti all'infinito" by U. Cassina, 326–337; "Il teorema Descartes-Eulero relativo ai poliedri" by A. Maroni, 337–346; "La questione dei nove valori nella risoluzione della cubica" by C. Candido, 347–351; "Il concetto delle due divisioni nell'insegnamento elementare e normale" by G. Furlani, 351–354; "Polemica logico-matematica C. Burali-Forti e F. Enriques," 354–365—Volume 2, January, 1922: "Sui meccanismi articolati nelle costruzioni di Geometria elementare" by E. G. Togliatti, 41–49; "L'area di una superficie curva nella definizione di Minkowski e nell'insegnamento della Geometria elementare" by L. Brusotti, 49–55; "Una rapida visione geometrica del teorema fondamentale dell' Algebra" by O. Chisini, 55–59; "Sui massimi e minimi e sulla variazione di una funzione reale" by E. Maccaferri, 60–67; "Su la configurazione dei circoli dei nove punta dei triangoli determinati da un gruppo di punti situati su una medesima circonferenza"

by D. Montesano, 68–70; "Einstein e l'interpretazione subiettiva della scienza," 77–80 [Address of F. Enriques to the public of Bologna on the occasion of the conference which Einstein held there in October, 1921]; "Atti della Società Italiana di Matematiche 'Mathesis,' Ottobre 1921," 90–116.

POPULAR ASTRONOMY, volume 30, March, 1922: "Suggestions regarding gravitation" by W. H. Pickering, 155–156; "Note on the velocity of light" by H. Shapley, 192–193 [By an extremely delicate test of the relative velocity of light of different colors it is shown that blue and yellow light can travel through space for 40,000 years without a difference of more than one or two minutes, if indeed there is any difference whatever]—April: "Henrietta Swan Leavitt" by S. I. Bailey, 197–199 [With photograph. "Miss Leavitt graduated from Radcliffe College in 1892. Later she spent several years as advanced student and volunteer research assistant at the Harvard Observatory, in travel, and in teaching. She became a permanent member of the Observatory staff in 1902, and thereafter was closely associated for the remainder of her life with Prof. Pickering, in the solution of problems especially connected with the determination of the photographic magnitudes of stars. Beginning work as an ordinary assistant in the measurement of variable stars on photographic plates, by her exceptional ability and intense application she soon became the head of the department of photographic stellar photometry. . . . Miss Leavitt was of an especially quiet and retiring nature, and absorbed in her work to an unusual degree. She had the highest esteem of all her associates at the Harvard Observatory, where her loss is keenly felt."]; "Shall we accept relativity?" by W. H. Pickering, 199–203 ["Three years have now elapsed since the Principle of Relativity was first announced to a startled public, as perhaps the greatest scientific discovery since the time of Newton. Little attention had been paid to it before that, outside of the ranks of professional physicists. Now that the glamour has somewhat subsided, it may be a good time to take account of stock, to see what evidence has meanwhile accumulated in its favor, and how far its supporters have increased among scientific men. Regarding the latter, the physicists it is believed support it as before. Among the astronomers its stronghold is undoubtedly in England. In America opinion is divided. Which way the majority vote would be cast it is impossible to say. On the continent of Europe I have found the Latins generally skeptical, while I understand that very few of the Germans take it seriously at the present time. Three years is of course too brief a time for any important discovery to win universal acceptance in science. Perhaps a generation may be taken as a fair interval, sometimes a little less, sometimes a little more. Therefore if its adherents have not increased in the past three years, or have even fallen off, that can hardly be counted very seriously against it. Let us now see what evidence has accumulated with regard to it in the meantime. . . . Summarizing our conclusions, we may say that in the first test, . . . the theory of relativity fails to give results in accord with the facts. Regarding the second test, the only result so far obtained is that it has now been clearly and satisfactorily demonstrated what not to do. The evidence so far regarding the third test is as we all know, and have seen, distinctly adverse."]; "Cosmic clouds" by W. F. Rigge, 203–205; "How to know the heavens" by G. B. Chase, 210–213; "The duration of sunrise and sunset" by W. J. Fisher, 213–216; "A letter of John Kepler" by F. Cajori, 217–219; "Planetary configurations" by F. R. Honey, 219–222; "Effect of the sun-spots on the terrestrial temperature" by P. G. Searles, 222–225; "Hypothesis on the formation of new stars" by A. Verronet, 225–227—May: "Eratosthenes—III" by W. H. Pickering, 257–262; "What about Mars?" by J. G. Porter, 268–271; "A suggestion regarding gravitation, II" by W. H. Pickering, 272; "'Astronomy improved'" by C. Rufus, 272–280; "The secular comets and the movement of the sun through space" by G. Armellini, 280–286; "The present position of the island universe theory of the spiral nebulæ" by D. B. McLaughlin, 286–295 (to be continued).

REVUE GÉNÉRALE DES SCIENCES, volume 33, February 15, 1922: "Camille Jordan" by R. d'Adhémar, 65–66 [see *1922*, 139]—February 28: Review by M. Solovine of J. M. Child's *Early Mathematical Manuscripts of Leibniz*, 121.

REVUE DE MATHÉMATIQUES SPÉCIALES, volume 32, January, 1922: "Sur les conditions nécessaires et suffisantes pour que l'hyperboloïde défini par trois droites soit de révolution" by M. Chenevier, 73–76—February: "Sur les sommes des puissances semblables des nombres entiers" by M. Weber, 97–100—March: "Note sur la balance" by M. Colin, 121–123—April: "Note sur la transformation conforme" by M. Lapointe, 145–146.

REVUE SCIENTIFIQUE, volume 59, November 12, 1921: "Le mois mathématique à l'Académie des Sciences (juillet et août 1921)" by R. Garnier, 620–621 [G. Bertrand suggests a formula that conforms to the law of Einstein for the displacement of the perihelion of a planet; E. Borel announces explicitly four fundamental hypotheses which connect the general principles of physics with those of geometry in the theory of relativity; K. Ogara gives an explanation for

the element of integral around a center in the theory of gravitation of Einstein]; "La mécanique classique et la théorie de la relativité" by P. Painlevé, 621–622; "Quelques remarques sur la théorie de la relativité" by E. Picard, 622–623—Volume 60, January 14, 1922: "La durée et la conception Einsteinienne du temps" by R. Paucot, 7–17—January 28: "A propos d'un projet de Calendrier mensuel fixe" by G. Galleano, 55–56—February 11: "Camille Jordan" by R. S., 95–96 [contains the address of E. Picard at the Academy of Sciences on January 23 (see *1922*, 139)] —March 25: "L'histoire de la nomographie" by S. R., 197–198 [Extract from *Comptes Rendus de l'Académie des Sciences* for January 9, page 82.]

SCHOOL SCIENCE AND MATHEMATICS, volume 22, January, 1922: "Class exercise types in high school mathematics, with norms for judging them. II" by G. W. Myers, 7–15; "Required mathematics in the four year high school" by R. D. Shouse, 15–19—February: "Square root of a line without use of the circle" by J. B. Wood, 111–113; "An analysis of an experiment in teaching first year mathematics" by Ina E. Holroyd, 114–121; "Learning fractions" by Myrtie Collier, 121–127; "Problem department" by J. A. Nyberg, 176–180; "Annual meeting of the Central Association of Science and Mathematics Teachers" by G. Warner, 190, 192, 194—March: "The fourth dimension: An explanation the geometry class can follow" by J. V. Collins, 226-231; "The law of exponents" by R. Morris, 232–236; "New mathematical periodicals" by G. A. Miller, 276–280; "Problem department" by J. A. Nyberg, 282–284, 286.

SCIENCE, new series, volume 55, January 6, 1922: "Henry Turner Eddy" by J. J. F., 12–13 —February 10: "Subsidy funds for mathematical projects" by H. E. Slaught, 146–148—February 17: "Relativity and star diameters" by R. A. Fessenden, 180—February 24: "A mechanical analogy in the theory of equations" by D. R. Curtiss, 189–194—March 17: "Doctorates conferred in the sciences by American universities in 1921" by Callie Hull and C. J. West, 271–279.

SPHINX-ŒDIPE, volume 16, November, 1921: "Notice sur Charles Ange Laisant" (continued) by H. Brocard, 161–166; "Sur les racines primitives relatives au mod. P^n dans un corps algébrique quelconque" (concluded) by G. Métrod, 166–168; Questions and Responses, 169–176; "Divers," 176 ["M. R. C. Archibald visitera dans 6 mois des universités européennes. Meilleurs voeux en attendant sa venue parmi nous."]—December: "Notice sur Charles Ange Laisant" (concluded) by H. Brocard, 177–182; Questions and Responses, 183–189; "Table des matières de 1921," 190–192—Volume 17, January, 1922: "Triangles et tétraèdres" by V. Thébault, 2–3; "Histoire des sciences: Correspondance de J. J. Sylvester—Ed. Lucas" translated by L. Chanzy, 4–7 (to be continued); "Solution de l'equation $4\alpha^4 - 4\alpha^3\beta + 13\alpha^2\beta^2 - 36\alpha\beta^3 + 24\beta^4 = \beta^2$" by L. Aubry, 7–10; Questions and Responses, 10–16—February: "Sur trois triangles homothétiques" by V. Thébault, 17–19; "Note sur un problème de la théorie des nombres d'Ed. Lucas" by L. Aubry, 20; Questions and Responses, 21–32.

TÔHOKU MATHEMATICAL JOURNAL, volume 20, nos. 1–2, December, 1921: "On the roots of an algebraic equation $f + k_1 f' + k_2 f'' + \cdots + k_n f^{(n)} = 0$" by K. Ôishi, 1–17; "Über die Gültigkeitsbedingung der Interpolationsformeln von Gauss" by M. Fujiwara, 18–21; "Sur les différences de deux ensembles fermés" by C. Kuratowski and W. Sierpínski, 22–25; "Ueber die Coefficienten in der Taylorschen Entwicklung rationaler Functionen" by C. Siegel, 26–31; "Tautochronous motion" by P. P. Upadhyaya, 32–36; "Criteria for the irreducibility of a polynomial" by K. Ôishi, 37–40; "Sur la configuration des circonférences des neuf points des triangles determinés par un groupe de points d'une même circonférence" by M. D. Montesano, 41–43 [The printers have somewhat disarranged the parts of this article. If the reader will divide the article into four nearly equal parts and read the third part before he reads the second part, he will get the correct arrangement of paragraphs as may be seen by comparison with essentially the same article printed in Italian in *Periodico di Mathematica* for January (see above).]; "Some theorems on infinite products" by C. Kien-Kwong, 44–47; "Ein Problem in der Wahrscheinlichkeitsrechnung" by M. Fujiwara, 48–50; "Über Stützgeradefunktion der konvexen geschlossenen Kurven" by M. Fujiwara, 51–59; "Sur deux théorèmes de Casey" by V. Thébault, 60–63; "On the representations of functions by the formulas of interpolation" by Y. Okada, 64–99; "On the definite integral $\int_a^\infty \varphi(t)e^{-zt}dt$" by K. Kurosu, 100–106; "On the integrals

$$\int_0^\pi \exp\left(x \begin{matrix}\cos\\ \sin\end{matrix} p\theta\right) \begin{matrix}\cos\\ \sin\end{matrix} q\theta d\theta"$$

by T. Hayashi, 107–114.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, Williams College, Williamstown, Mass.

CLUB ACTIVITIES.

AGNESI CLUB OF AGNES SCOTT COLLEGE, Decatur, Ga.

The Agnesi Club was formed at Agnes Scott College, November 28, 1921, at the suggestion of the Departments of Mathematics and Physics. Students in the elective courses of mathematics, astronomy, and physics are eligible for membership. The Club has twenty members. Meetings are held monthly. The officers elected at the first meeting were: President, Mary Barton '22; vice-president, Elizabeth Hoke '23; secretary, Emmie Ficklen '24.

December 12, 1921: "Radium and the X-ray" by Professor Housen; "Agnesi" by Elizabeth Hoke '23. Adoption of the Constitution.

(Report by Miss Ficklen.)

MATHEMATICS CLUB OF BUCKNELL UNIVERSITY, Lewisburg, Pa.

The Mathematics Club of Bucknell University was organized January 28, 1920, through the efforts of Professor H. S. Everett. Active members must have passed all required mathematics, also analytical geometry and one other subject in pure or applied mathematics. Associate membership is open to anyone interested in mathematics. The following officers were elected: President, Sydney Peale '20; vice-president, Elizabeth Spyker Gr.; secretary-treasurer, Grace Poust '22. The following papers were presented:

January 28, 1920: "Non-Euclidean world" by Professor W. C. Bartol.

February 11: "How x came to be used as an unknown" by Lillian Russell '23; "Two applications of the spiral of Archimedes" by Robert Haberstroh '23.

February 25: "Trisection of an angle" by Irvin Holmes '20.

March 10: "Trisection of an angle—continued" by Irvin Holmes '20; "The quadrature of the circle" by Clara Casner '21; "The duplication of the cube" by Elizabeth Weidner '21.

April 7: "Famous mathematicians" by Professor Everett.

April 26: "Probabilities" by Sydney Peale '20.

For the year 1920–21, the following officers were elected: President, Elizabeth Weidner '21; vice-president, Leonard Worthington '21; secretary-treasurer, Elsie Watson '21; critic, Professor J. S. Gold. Papers were presented during the year as follows:

November 1, 1920: "Magic squares" by Professor C. A. Lindeman, of the Department of Engineering.

November 15: "Paper folding" by Kathryn Kimble '23; "Something about zero" by Grace Fry '22.

December 6: "Number puzzles" by Professor Gold.

January 17, 1921: "The planimeter" by Chester Derck '22; "Mechanism for solving a cubic" by Paul Mallay '23.

February 14: "Diophantine analysis" by George Lowry '20.

February 28: "Numerical results for diverse systems of breeding" by Professor F. N. Davis, of the Department of Biology.

March 14: "Use of logarithmic paper" by Leonard Worthington '21.

April 11: "History of codes and ciphers" by Clara Casner '21; "Non-Euclidean geometry" by Professor Everett.

April 18: "Codes and ciphers" by Elsie Watson '21; "A play—The Flatlanders"; "The fourth dimension" by Professor Everett.

April 25: "Einstein's theory of relativity" by Professor Everett.

For the year 1921–22, the following officers were elected: President, Chester Derck '22; vice-president, Grace Poust '22; secretary-treasurer, Kathryn Kimble '23. Programs were presented as follows:

October 10, 1921: "Sir Isaac Newton" by Professor Bartol.

November 14: "Euclid's life and works" by Grace Poust '22; "Archimedes' life and work" by Chester Derck '22; "Life and work of Apollonius" by Ethylwynne Smith '23.

December 12: "Mathematics in colonial times" by Leona Dickrager '22; "Alice in the Wonderland of mathematics" by Nina Grace Smith '23; "Numerical curiosities" by Philip Campbell '22.

January 16, 1922: "Cardan's solution of the cubic" by Professor Lindeman.

February 13: "Life and works of Descartes" by Ruth King '22; "Life and works of Newton" by Nellie Balliet '22.

March 13: "Projective geometry" by Professor Gold.

The number of members each year has been about 40; and the average attendance about 25. In the fall and in the spring also there is a social party.

(Report by Miss Kimble.)

The Mathematics Club of Goucher College, Baltimore, Md.

[*1918*, 357; *1919*, 365; *1921*, 274; *1922*, 26.]

Meetings during the second half of 1920–21 were held as follows:

March 3: "Human calculating prodigies" by Grace McCaulley '23; "The theory of Amsler's planimeter" by Katherine Wisner '21.

April 7: "A new theorem concerning three circles" by Henrietta Morris '23; "Proof that an angle cannot be trisected by ruler and compass" by Eva Lazarus '22.

April 24: "History of the theory of numbers, with illustrative problems" by Dorothy Biscoe '22; "The fallacy in a recently proposed method of trisecting an angle by ruler and compass" by Mildred Brown '21; "Rules for checking addition, with proofs" by Mildred Trueheart '22.

The annual picnic of the Club took place in May, and was attended by many of its alumnæ members.

(Reported by Professor Lewis.)

The Mathematics Club of Vassar College, Poughkeepsie, N. Y.

[*1918*, 136, 456; *1919*, 264; *1920*, 78, 427.]

October 21, 1920: A tea was given, to which were invited all who were eligible to membership. The following officers were elected for the first semester: President, Mary West '21; vice-president, Theresa McMakin '22; secretary-treasurer, Alta Teeple '22; executive committee, Professor Elizabeth B. Cowley, Elizabeth Larsen '21.

November 16: "Women in mathematics" by Elinor Hollis '21.

December 9: "History, development, and use of logarithms" by Grace Fenstermaker '21, Mathematical tricks were exhibited by Elsie Stuart '22.

February 24, 1921: The following officers were elected: President, Amy Davison '21; vice-president, Theresa McMakin '22; secretary-treasurer, Alta Teeple '22; executive committee, Professor Cowley, Elizabeth Larsen '21.

March 17: "The Einstein theory" by Elizabeth Larsen '21.

April 28: "Codes and ciphers" by Dorothy Laws '22; "The fourth dimension" by Edith Morrison '22.

May 19: A picnic supper. Charades were played with such words as hypothenuse, perpendicular, and algebra.

(Reported by Miss Teeple.)

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. P. Manning.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible

sources will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2966. Proposed by OTTO DUNKEL, Washington University.

When a curve produces a caustic by the reflection of the rays proceeding from a fixed point, δ and δ' the lengths of the incident and reflected rays, R the radius of curvature of the curve, and ω the angle of incidence, satisfy the equation (see *1920*, 225)

$$\frac{1}{\delta}+\frac{1}{\delta'}=\frac{2}{R\cos\omega}.$$

Give a geometrical proof of this relation using the harmonic properties of the figure and obtain in this manner a simpler and different geometrical derivation from that given in Humbert's *Cours d'Analyse*, volume 1, page 77.

2967. Proposed by ELIJAH SWIFT, University of Vermont.

A plane revolves about one of two non-coplanar lines as an axis. Find the locus, *in the plane*, of the intersection of the plane and the other line.

2968. Proposed by MALCOLM FOSTER, Yale University.

A curve C is the directrix of a ruled surface, and g is any ruling. Relative to the trihedral of C at the point of intersection with g, the direction-cosines of g are α, β, γ, expressed as functions of the arc of C. Prove that the distance t along g from C to the line of striction is given by

$$t=\frac{\dfrac{\beta}{\rho}-\alpha'}{\Sigma\alpha'^2+\dfrac{2}{\rho}(\alpha\beta'-\alpha'\beta)+\dfrac{2}{\tau}(\beta'\gamma-\beta\gamma')+\dfrac{1}{\rho^2}(1-\gamma^2)+\dfrac{1}{\tau^2}(1-\alpha^2)+\dfrac{2}{\rho\tau}\alpha\gamma},$$

where ρ and τ are the radii of first and second curvature of C.

2969. Proposed by J. N. VEDDER, Union College, Schenectady, N. Y.

The following integral equations arise in connection with electrical theory:

$$f(x)=c-\lambda\int_{-a}^{+a}\log\left|\frac{a-u}{x-u}\right|f(u)du \tag{1}$$

and

$$f(x,y)=c-\lambda\int_{-a}^{+a}\int_{-b}^{+b}\log\sqrt{\frac{(a-u)^2+(b-v)^2}{(x-u)^2+(y-v)^2}}f(u,v)du\,dv. \tag{2}$$

The value of $\int_{-a}^{+a}f(x)dx$ is required from (1) and $\int_{-a}^{+a}\int_{-b}^{+b}f(x,y)dxdy$ from (2) in a form adapted to arithmetical calculation. In each case c is a given constant.

2970. Proposed by E. T. BELL, University of Washington.

For what values of the positive rational integers a, b, c, d, e, f are $a\cos^b(2\pi/c)$, $d\sin^e(2\pi/f)$ quadratic integers?

2971. Proposed by C. E. HORNE, The University of Porto Rico, Mayaguez, P. R.

Prove that the two tangents drawn to an ellipse from any external point subtend equal angles at a focus. (This problem is found in some text-books, but the Proposer is anxious to see an analytic solution of it.)

SOLUTIONS.

2828 [1920, 186]. Proposed by T. M. BLAKSLEE, Ames, Iowa.

On page 72 of R. B. Hayward's "*The Algebra of Coplanar Vectors and Trigonometry*" occurs the sentence: "It will be a good exercise for the student to show that $\cos(90°/7)=\frac{1}{2}\sqrt{x_1}$ where x_1 is the greatest root of the equation

$$x^3-7x^2+14x-7=0.$$

(1) Do not merely verify but deduce the equation and find x_1. (2) Deduce the x-equation (x_1, x_2, x_3, x_4 the roots) such that its greatest root x_1 gives $\cos(90°/9) = \cos 10° = \frac{1}{2}\sqrt{x_1}$. (3) Of what angles are $\frac{1}{2}\sqrt{x_1}, \cdots \frac{1}{2}\sqrt{x_4}$ in (2) the cosines? Develop a method of writing out at once $\cos nv$ in terms of powers of $\cos v$ if these are given for $(n-1)v$ and $(n-2)v$. The same for $\sin nv$. (4) Use the results of (2) and (3) to find the number of degrees in a radian. Hence, find π from radian instead of radian from π as is usual.

Solution by J. B. Reynolds, Lehigh University.

From the formula

$$\cos x + \cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y),$$

letting $x = nv$ and $y = (n-2)v$, we have[1]

$$\cos nv = 2\cos v\cos(n-1)v - \cos(n-2)v.$$

So also $\sin nv = 2\cos v\sin(n-1)v - \sin(n-2)v$.
Putting $n = 2, 3, \cdots 9$, we readily obtain

$$\cos 7v = 64\cos^7 v - 112\cos^5 v + 56\cos^3 v - 7\cos v,$$

and

$$\cos 9v = 256\cos^9 v - 576\cos^7 v + 432\cos^5 v - 120\cos^3 v + 9\cos v.$$

Replacing in these $\cos v = \frac{1}{2}\sqrt{x}$, we get

$$\cos 7v = \frac{x^{1/2}}{2}(x^3 - 7x^2 + 14x - 7), \tag{1}$$

$$\cos 9v = \frac{x^{1/2}}{2}(x^4 - 9x^3 + 27x^2 - 30x + 9). \tag{2}$$

If in (1), we let $\cos 7v = 0$ or $7v = (n+\frac{1}{2})\pi$, we have $v = \pi/14 = 90°/7$ for the greatest root $4\cos^2 v$, of

$$x^3 - 7x^2 + 14x - 7 = 0. \tag{3}$$

Again, if we let $\cos 9v = 0$ or $9v = (n+\frac{1}{2})\pi$, we have $v = (2n+1)\pi/18$ for the roots of

$$x^{1/2}(x^4 - 9x^3 + 27x^2 - 30x + 9) = 0; \tag{4}$$

and if we call them $x_1, x_2, x_3, x_4, 0$, for $n = 0, 1, 2, 3, 4$, then

$$x_1 = 4\cos^2\pi/18, \quad x_2 = 4\cos^2\pi/6, \quad x_3 = \cos^2 5\pi/18, \quad x_4 = 4\cos^2 7\pi/18, \quad x_5 = 4\cos^2\pi/2 = 0.$$

Since $4\cos^2\pi/6 = 3$, we may factor (4) into

$$x^{1/2}(x-3)(x^3 - 6x^2 + 9x - 3) = 0. \tag{5}$$

We may solve (3) and (5) by Horner's method to get any of the roots to the desired degree of accuracy, since the roots are all real. In this manner, we find from (5) $x_1 = 3.87938$, $x_3 = 1.65271$, and since $x_1 + x_3 + x_4 = 6$, $x_4 = .46791$, $x_2 = 3$. Now we have $\cos 70° = \frac{1}{2}\sqrt{x_4}$ or

$$70° = \cos^{-1}(\tfrac{1}{2}\sqrt{x_4}) = 90° - \sin^{-1}(\tfrac{1}{2}\sqrt{x_4}),$$

giving

$$20° \text{ in radians} = \sin^{-1}\left(\frac{x_4^{1/2}}{2}\right) = \frac{x_4^{1/2}}{2}\left\{1 + \frac{x_4}{24} + \frac{3x_4^2}{640} + \frac{5x_4^3}{64\cdot 112} + \cdots\right\}$$

$$= .349058 \text{ radians};$$

whence, 1 radian $= 57.297°$ or $\pi/9 = .349058$, giving $\pi = 3.14152$.

2830 [1920, 227]. Proposed by WILLIAM HOOVER, Columbus, Ohio.

An elastic string connects a pair of opposite vertices of a square whose sides are four equal rods freely jointed, each of length $2a$. The system is suspended vertically from a vertex attached to the string and is at rest. If the string be cut, and θ is the acute angle any side makes with the vertical at any moment during the motion, determine the angular velocity of any rod.

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[1] Compare Note 3, in this MONTHLY, *1921*, 38–39.

SOLUTION BY THE PROPOSER.

Change the word square in the problem to rhombus. Take the horizontal line through the fixed vertex for the x-coordinate axis, and the vertical diagonal of the rhombus formed of the rods for the y-axis. Let x_1, y_1 be the numerical values of the coordinates of each of the centers of the upper pair of rods; x_2, y_2 of each of the lower pair; m the mass of each rod; and α the value of θ upon the cutting of the string.

We have $x_1 = a \sin\theta$, $y_1 = a\cos\theta$; $x_2 = a\sin\theta$, $y_2 = 2a\cos\theta + a\cos\theta = 3a\cos\theta$.

The energy equation for the motion is

$$2\cdot\tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \frac{a^2}{3}\dot{\theta}^2\right) + 2\cdot\tfrac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2 + \frac{a^2}{3}\dot{\theta}^2\right) = 2mgy_1 + 2mgy_2 + C;$$

but $\dot{x}_1 = \dot{x}_2 = a\cos\theta\cdot\dot{\theta}$, $\dot{y}_1 = -a\sin\theta\cdot\dot{\theta}$, $\dot{y}_2 = -3a\sin\theta\cdot\dot{\theta}$; then

$$m\frac{4a^2}{3}\dot{\theta}^2 + ma^2\left(\frac{4}{3} + 8\sin^2\theta\right)\dot{\theta}^2 = 8mag\cos\theta + C.$$

But $\theta = \alpha$ when $\dot{\theta} = 0$; then $C = -8mag\cos\alpha$, and the required angular velocity is given by

$$a(1 + 3\sin^2\theta)\dot{\theta}^2 = 3g(\cos\theta - \cos\alpha) \qquad \text{or,} \qquad \dot{\theta} = \sqrt{\frac{3g(\cos\theta - \cos\alpha)}{a(1 + 3\sin^2\theta)}}.$$

Also solved by J. B. REYNOLDS and F. L. WILMER.

2836 [1920, 273]. Proposed by W. V. N. GARRETSON, Rutgers College.

A ladder 40 feet long rests with one end on the ground against the foot of a building and the other end against the side of a second building directly across the street from the first. A second ladder 25 feet long inclines in a similar manner from the foot of the second building against the side of the first building, the two ladders crossing at a point 15 feet above the ground. How wide is the street?

SOLUTION BY H. S. UHLER, Yale University.

Let x and y denote respectively the width of the street and the distance from the foot of the first building to the lowest point of the 15 ft. vertical.

From similar right triangles, we obtain immediately

$$\frac{15}{\sqrt{1600 - x^2}} = \frac{y}{x}$$

and

$$\frac{15}{\sqrt{625 - x^2}} = \frac{x - y}{x} = 1 - \frac{y}{x}.$$

Therefore, by addition,

$$\frac{15}{\sqrt{1600 - x^2}} + \frac{15}{\sqrt{625 - x^2}} - 1 = 0. \tag{1}$$

The solution of equation (1) may be effected advantageously in the following manner. Let the second fraction be represented by $1/u$ so that

$$x = 5\sqrt{25 - 9u^2}, \tag{2}$$

and equation (1) transforms into

$$\frac{\sqrt{3}}{\sqrt{13 + 3u^2}} = \frac{u - 1}{u},$$

or

$$3u^4 - 6u^3 + 13u^2 - 26u + 13 = 0.$$

Solving the last equation by Horner's method of approximation we obtain $u = 1.612347811$. The required result is now found by substituting in relation (2). It is $x = 6.33050319$ ft.

Also solved by T. M. BLAKSLEE, E. B. ESCOTT, LAURA GUGGENBUHL, W. W. JOHNSON, G. A. KNAPP, R. M. MATHEWS, H. L. OLSON, ARTHUR PELLETIER, J. B. REYNOLDS, and C. C. WYLIE.

2839 [1920, 274].

By translating the steps of the construction of a regular pentagon from plane geometry into algebra show that one of the fifth roots of unity is equal to

$$\tfrac{1}{4}(\sqrt{5}-1)+\tfrac{1}{4}i\sqrt{10+2\sqrt{5}}.$$

(This problem is proposed for solution in Wilczynski and Slaught, *College Algebra with Applications*, Boston, 1916, p. 193.)

I. Solution by H. L. Olson, University of Michigan.

Consider a circle with center at the origin, O, and unit radius, intersecting the positive x-axis at A and the negative y-axis at D. With center at the mid-point, E, of OD, draw a circular arc passing through A and intersecting the positive y-axis at F. The radius of this circle is $\frac{1}{2}\sqrt{5}$, and hence $OF = \frac{1}{2}(\sqrt{5}-1)$. With center A and radius AF, draw a circular arc intersecting the circle with center O at G and H. AG and AH are two sides of a regular pentagon inscribed in the circle O, and $\overline{OG}$ and $\overline{OH}$ are the Argand representations of two of the fifth roots of unity.

Since $AF^2 = 1 + (\frac{1}{2}\sqrt{5}-\frac{1}{2})^2$, the equations of the circles O and A are

$$x^2+y^2=1,$$
$$(x-1)^2+y^2=1+(\tfrac{1}{2}\sqrt{5}-\tfrac{1}{2})^2.$$

Hence, the coördinates of G and H are

$$x=\tfrac{1}{4}(\sqrt{5}-1), \qquad y=\pm\tfrac{1}{4}\sqrt{10+2\sqrt{5}};$$

and two of the fifths roots of unity are $\frac{1}{4}[(\sqrt{5}-1)\pm i\sqrt{10+2\sqrt{5}}]$.

II. Solution by Otto Dunkel, Washington University.

In texts on geometry a pentagon is usually constructed by dividing the radius OA, here taken as of unit length, at M in extreme and mean ratio. The length OM is laid off twice on the circle as chords giving the points A, K, G; then AK is the side of a regular decagon and AG is the side of a regular pentagon. From the definition of extreme and mean ratio it follows that $OM^2 = OA\cdot MA = 1-OM$, and hence $OM=\frac{1}{2}(\sqrt{5}-1)$. It easily follows that $MG=OG=1$ and hence the coördinates of G are $x=\frac{1}{4}(\sqrt{5}-1)$, $y=\sqrt{1-x^2}=\frac{1}{4}\sqrt{10+2\sqrt{5}}$. Hence the complex number represented by G is as stated in the problem.

Also solved by T. M. Blakslee, Arthur Pelletier, A. V. Richardson, C. H. Richardson, and F. L. Wilmer.

2853 [1920, 377]. Proposed by J. S. Brown, Southwest Texas State Normal College, San Marcos, Texas.

Find the side and apothem of a regular pentagon inscribed in a circle without the use of extreme and mean ratio.

Three Solutions by T. M. Blakslee, Ames, Iowa.

I. Let $x+iy$ be the point on the unit circle whose angle is 36°. Then the side of a regular inscribed pentagon will be $p=2y$. In the equation $(x+iy)^5=-1$, the coefficient of i is

$$y(5x^4-10x^2y^2+y^4)=0.$$

We wish the smaller of the two positive values of y. Therefore we can remove the factor y and if we substitute $1-y^2$ for x^2, our equation reduces to

$$16y^4-20y^2+5=0.$$

Whence $y^2=(10-2\sqrt{5})/16$, $p=\frac{1}{2}\sqrt{10-2\sqrt{5}}$, and the apothem is $a=\frac{1}{4}(\sqrt{5}+1)$.

II. $p = 2 \sin 36° = 4s\sqrt{1 - s^2}$, where $s = \sin 18° = \cos 72°$. But $72°$ is four times $18°$; therefore s is a root of the equation

$$8s^4 - 8s^2 - s + 1 = 0.$$

The factors $s - 1$ and $2s + 1$ correspond to the angles $90°$ and $-30°$. Thus we have $4s^2 + 2s - 1 = 0$ and the positive root is $s = \frac{1}{4}(\sqrt{5} - 1)$, from which we get p and a.

III. Let d be the side of the regular decagon inscribed in the same circle; then d being a chord of an arc in a unit circle and p that of twice the same arc,

$$d^2 = 2 - \sqrt{4 - p^2}.$$

Let O be the center of the circle, AB and AC chords equal to p and d, F the foot of the perpendicular from A upon OB, and G the middle point of AB. The four points O, F, G and A are concyclic and $BA \cdot BG = BO \cdot BF$.

Now the triangle AOF is congruent to the right triangle formed by the radius and apothem of the decagon, having the same angles and hypothenuse equal to 1. Therefore $OF = d/2$ and our equation may be written

$$p \cdot \frac{p}{2} = 1\left(1 - \frac{d}{2}\right) \text{ or } p^2 = 2 - d.$$

Eliminating d and removing the factor $p^2 - 3$, we have

$$p^4 - 5p^2 + 5 = 0$$

or, since p is only a very little more than 1, $p^2 = \frac{1}{2}(5 - \sqrt{5})$. This gives p and then a as above.

Also solved by MICHAEL GOLDBERG, R. M. MARSHALL, J. Q. MCNATT, ARTHUR PELLETIER, A. V. RICHARDSON, and L. S. SHIVELY. Some have interpreted the problem as calling for a geometrical construction rather than an algebraic solution.

2859 [1920, 428]. Proposed by L. S. DEDERICK, U. S. Naval Academy.

Derive an expression for the limit of error in evaluating a definite integral by Simpson's Rule.

I. SOLUTION BY A. A. BENNETT, University of Texas.

Let the desired definite integral be $\int_{a-h}^{a+h} f(x)dx$, where $h > 0$. If the function $f(x)$ be supposed capable of a Taylor expansion through the term in $(x - a)^3$, with a remainder, a simple solution for the problem is obtained by employing this expansion, the form of the result depending upon the form of the remainder adopted.

The problem as stated should not depend upon the existence of derivatives at any point of the interval. A solution not involving such derivatives is the following:

$$\left| \int_{a-h}^{a+h} f(x)dx - \frac{h}{3}\{f(a+h) + 4f(a) + f(a-h)\} \right| \leqq \text{Max}\{h|x-a|^2|D(x) - D(a+h)|\}, \quad (1)$$

where $(x - a)^2 D(x) \equiv f(x) - 2f(a) + f(2a - x)$.

If $f(x)$ has a second derivative at $x = a$, this is of course the value approached by $D(x)$ as x approaches a. To prove (1), write

$$g(x) \equiv h(x - a)^2\{D(x) - D(a + h)\}.$$

Expanding and integrating we have

$$\int_{a-h}^{a+h} g(x)dx = 2h\left[\int_{a-h}^{a+h} f(x)dx - \frac{h}{3}\{f(a+h) + 4f(a) + f(a-h)\}\right].$$

Since $\left| \int_{a-h}^{a+h} g(x)dx \right| \leqq 2h \text{ Max } |g(x)|$, (1) is proved.

It is to be noted that $\text{Max}\{h|x - a|^2|D(x) - D(a + h)|\} = 0$, for polynomials of less than the fourth order, so that the formula given by Simpson's rule is exact in these cases.

II. Note on Professor Bennett's solution by H. P. Manning, Providence, R. I., and Otto Dunkel, Washington University.

If $f(x)$ can be expanded to four terms with the Lagrange remainder, the substitution of this expansion will show that $\max |g(x)| \leqq \frac{h^5}{3!} \max |f^{iv}(x)|$, but substitution in the integral $\int_{a-h}^{a+h} g(x)dx$ gives for the error the smaller limit $\frac{2}{45} h^5 \max |f^{iv}(x)|$, the same that is given by the substitution for $f(x)$, and for $f(a+h)$ and $f(a-h)$ in Simpson's formula itself. This limit is given in E. B. Wilson's *Advanced Calculus*, 1912, pages 76–77, exercises 23 and 24. However, a limit still smaller, namely, $\frac{h^5}{90} \max |f^{iv}(x)|$, has been found and is given by P. J. Daniell in this Monthly, *1917*, 110, and also by C. J. de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, volume 1, third edition, 1914, page 396. The method employed by the latter can be applied to the integral $\int_{a-h}^{a+h} g(x)dx$ and leads to the same result. Thus Professor Bennett's expression, obtained without assuming any expansion, leads to the results already found for functions capable of expansion to four terms and a remainder.

In 1874 Chevilliet (*Comptes Rendus de l'Académie des Sciences*, volume 78, page 1841), by taking the infinite expansion of $f(x)$, shows that the first term of the error is $-\frac{h^5}{90} f^{iv}(a)$, and when h is sufficiently small this approximates to the limit given by de la Vallée-Poussin. This result is obtained in the same way in Kiepert's *Grundriss der Differential- und Integralrechnung*, Teil 2, seventh edition, 1900, pages 335–336.

In Heine's *Handbuch der Kugelfunktionen*, Band 2, Theil 1, 1881, there is an exhaustive discussion of mechanical quadrature, and expressions are obtained for the errors in the Newton-Cotes method and in the method of Gauss. In particular, the fraction $-1/90$ can be obtained by multiplying $1/4!$ by the $-4/15$ given in the table on page 9.

William Hoover gave the reference to Wilson's *Calculus*, and H. E. Jordan to Kiepert's work.

2868 [1920, 482]. Proposed by H. S. Uhler, Yale University.

Let the evolute of a given curve be called the evolute of the first order, let the evolute of the first evolute be called the evolute of the second order, etc. Then, being given the following parametric equations in which a is a constant and γ is the parameter, namely,

$$x = (1 + 2\sin^2\gamma)\cos\gamma - a\sin 2\gamma, \qquad y = 2\sin^3\gamma + a\cos 2\gamma,$$

find: (a) the parametric equations of the evolute of order n, both for n even and for n odd;
(b) a formula for the total length of the nth evolute;
(c) a formula for the total area of the nth evolute;
(d) the sum of the lengths of all the evolutes from $n = 1$ to $n = \infty$; and
(e) the sum of the areas of all the evolutes from $n = 1$ to $n = \infty$.

Note. The original equations represent the envelope required in problem 2819 (*1920*, 134).

I. Solution by F. L. Wilmer, Omaha, Neb., and H. P. Manning, Providence, R. I.

One may note that the equation of the normal to the given curve can be put into the p-form with p a simple function of the parameter. Then differentiation with respect to the latter must give the equation of a perpendicular meeting this line in the corresponding point of the evolute, and so normal to the latter. In this way are obtained the parametric equations of the evolute, and by repetition those of the nth evolute.

We can write the given equations

$$\begin{aligned} x &= 3\cos\gamma - 2\cos^3\gamma - a\sin 2\gamma, \\ y &= \qquad 2\sin^3\gamma + a\cos 2\gamma; \end{aligned}$$

and from these it follows that $dy/dx = \tan 2\gamma$.

Now the equation of the normal will be

$$X \cos 2\gamma + Y \sin 2\gamma = x \cos 2\gamma + y \sin 2\gamma = \cos \gamma.$$

Thus for the first evolute we have the equations

$$x_1 \cos 2\gamma + y_1 \sin 2\gamma = \cos \gamma, \qquad x_1 \sin 2\gamma - y_1 \cos 2\gamma = (\sin \gamma)/2;$$

or

$$x_1 = \cos^3 \gamma, \qquad 2y_1 = 3 \sin \gamma - 2 \sin^3 \gamma.$$

Similarly, starting with these equations, we get for the second evolute[1]

$$2(x_2 \sin 2\gamma - y_2 \cos 2\gamma) = \sin \gamma, \qquad 2(x_2 \cos 2\gamma + y_2 \sin 2\gamma) = (\cos \gamma)/2;$$

or

$$2^2 x_2 = 3 \cos \gamma - 2 \cos^3 \gamma, \qquad 2y_2 = \sin^3 \gamma.$$

If we let x_0 and y_0 be what x and y become when a is zero, then $2^2 x_2 = x_0$ and $2^2 y_2 = y_0$.

Suppose

$$2^{n-1} x_n = \cos^3 \gamma, \qquad 2^n y_n = 3 \sin \gamma - 2 \sin^3 \gamma; \tag{1}$$

so that $2^{n-1} x_n = x_1$ and $2^{n-1} y_n = y_1$. Then we shall have

$$2^{n+1} x_{n+1} = 3 \cos \gamma - 2 \cos^3 \gamma, \qquad 2^n y_{n+1} = \sin^3 \gamma,$$

or $2^{n+1} x_{n+1} = x_0$ and $2^{n+1} y_{n+1} = y_0$, getting these equations in the same way that the equations for x_2 and y_2 were derived from those for x_1 and y_1.

Again, suppose

$$2^n x_n = 3 \cos \gamma - 2 \cos^3 \gamma, \qquad 2^{n-1} y_n = \sin^3 \gamma; \tag{2}$$

so that $2^n x_n = x_0$ and $2^n y_n = y_0$. Then $2^n x_{n+1} = \cos^3 \gamma$, and $2^{n+1} y_{n+1} = 3 \sin \gamma - 2 \sin^3 \gamma$, or $2^n x_{n+1} = x_1$ and $2^n y_{n+1} = y_1$.

Thus we prove by induction that (2) holds for n even and (1) for n odd.

For lengths we have $ds_0/d\gamma = 3 \sin \gamma$, $2ds_1/d\gamma = 3 \cos \gamma$; hence for n even $2^n ds_n/d\gamma = ds_0/d\gamma = 3 \sin \gamma$, and for n odd $2^n ds_n/d\gamma = 2ds_1/d\gamma = 3 \cos \gamma$. Integrating through 90° and multiplying by 4, we have in all cases for the complete length of the nth evolute $3/2^{n-2}$.

The length of the first evolute is 6 and the sum of the lengths of all of the evolutes is

$$6 + \sum_{n=2}^{\infty} 3/2^{n-2} = 12.$$

When n is even $ds_n/d\gamma$ becomes zero and changes sign for $\gamma = 0$ or π; therefore each of these evolutes has two cusps on the x-axis.

When n is odd $ds_n/d\gamma$ becomes zero and changes sign for $\gamma = \pm \pi/2$, and each of these evolutes has two cusps on the y-axis.

Areas can be obtained by the formula $A = 4\int_0^1 y dx = -4\int_0^{\pi/2} y \frac{dx}{d\gamma} d\gamma$.

Now $y_0 dx_0/d\gamma = -2 \sin^3 \gamma (3 \sin \gamma - 6 \cos^2 \gamma \sin \gamma)$; therefore $A_0 = 3\pi$.

Also $2y_1 dx_1/d\gamma = (3 \sin \gamma - 2 \sin^3 \gamma)(-3 \cos^2 \gamma \sin \gamma)$; therefore $A_1 = 3\pi/4$.

Then when n is even $2^n A_n = A_0$, and when n is odd $2^{2(n-1)} A_n = A_1$. Thus $A_n = 3\pi/2^{2n}$ for all values of n.

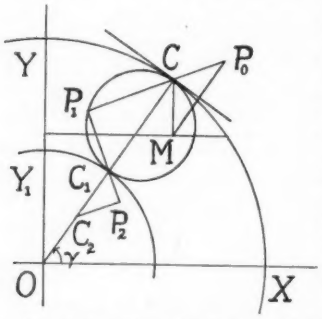
The sum of the areas of all of the evolutes is $\sum_{n=1}^{\infty} 3\pi/2^{2n} = \pi$.

Note. The area between two successive evolutes is swept over by the radius of curvature of the larger as γ varies through 360°. Thus we can write $A_n - A_{n+1} = \int_0^{2\pi} \frac{\rho_n^2}{2} \cdot 2d\gamma$, where ρ_n is the radius of curvature of the nth evolute. But then $\rho_n = s_{n+1} = (1/2^{n+1}) \cdot 3 \sin \gamma$ (or $3 \cos \gamma$ when n is odd). Thus this integral becomes $\frac{9}{2^{2(n+1)}} \int_0^{2\pi} \sin^2 \gamma$ (or $\cos^2 \gamma$) $d\gamma = \frac{9\pi}{2^{2(n+1)}}$.

[1] It is not necessary to go through the process of deriving the first of these two equations, since it is the same as the second of the two equations for x_1 and y_1, the normal to the first evolute being the tangent to the second.

II. Solution by Otto Dunkel, Washington University.

The solution of this problem, like that of 2819 (*1921*, 190) is simplified by the geometry of the curves. These may be constructed as follows: A unit circle is drawn with the origin as center and a chord parallel to the x-axis at a units above it. A radius OC is drawn with the inclination γ to the x-axis and the point C is projected upon the chord in the point M. Then the point P_0 which is symmetrical to M with respect to the tangent at C is a point on the given curve, P_0C is a normal, and the envelope of this normal, the first evolute of the present problem, is the caustic of the unit circle produced by vertical rays. If P_1 is the point of contact with the caustic, $CP_1 = (\sin\gamma)/2$ (formula (1), *1920*, 225, ω of the formula being equal to $\pi/2 - \gamma$ and R to 1), and this point is obtained by projecting the middle point C_1 of OC upon P_0C. P_1P_0 has the inclination $2\gamma - \pi/2$ and length $\rho_0 = (3/2)\sin\gamma - a$. Angle P_1CC_1 = angle YOC_1; therefore the arc P_1C_1 on the circle of diameter C_1C is equal to the arc Y_1C_1, where Y_1 is the point where the y-axis is cut by the circle with center O and radius $OC_1 = 1/2$. It follows that the locus of P_1 is the curve traced by this point when the former circle rolls on the latter circle; it is an epicycloid of two cusps, one at Y_1 and the other at the diametrically opposite point. This is our first evolute and P_1C_1 is its normal.



If $\rho_1 = P_1P_2$ is the radius of curvature of the first evolute, then $\rho_1 = (1/2)d\rho_0/d\gamma = 3(\cos\gamma)/4$, $C_1P_2 = (1/2)\,P_1C_1$, and P_2 is the projection upon this line of C_2 the middle point of OC_1. Now we prove as above for P_1, that the locus of P_2, our second evolute, is a two-cusped epicycloid, this time traced by rolling the circle of diameter C_2C_1 upon the fixed circle of radius OC_2, and having its cusps at the point where the latter circle cuts the x-axis and at the diametrically opposite point.

The same reasoning may be repeated again and again, giving us for the nth evolute an epicycloid with cusps on the x-axis when n is even and on the y-axis when n is odd. The equations in the former case are $x_n = (1/2^{n+1})(3\cos\gamma - \cos 3\gamma)$, $y_n = (1/2^{n+1})(3\sin\gamma - \sin 3\gamma)$, while the minus signs are changed to plus signs for the latter case.

The length of an arc of any evolute after the first measured from the nearest cusp of the preceding evolute is equal to the radius of curvature of the preceding evolute. Now $\rho_{n-1} = 3(\sin\gamma)/2^n$ or $3(\cos\gamma)/2^n$, neglecting signs; hence the complete length of the nth evolute can be obtained from one or the other of these expressions by putting the sine or cosine equal to 1 and multiplying by 4. That is, it is $3/2^{n-2}$ (it is 6 for $n = 1$).

For an evolute whose cusps lie on the y-axis the element of area generated by that portion of the radius of curvature which lies outside of the corresponding fixed circle is equal to twice the element $x\,dy$ for the circle. For the first evolute, for example, it is $(1/2)\cos^2\gamma\,d\gamma$. A similar relation, the axes being interchanged, holds for an evolute whose cusps lie on the x-axis. Therefore the area of any evolute of the system is 3 times the area of the corresponding circle; namely, for the nth evolute it is $3\pi/2^{2n}$.

Since the lengths and areas form geometrical progressions it is easy to find their sums.

NOTES AND NEWS.

It is hoped that readers of the Monthly will coöperate in contributing to the general interest of this department by sending items to H. P. Manning, Brown University, Providence, R. I.

Charles Leonard Bouton died at Cambridge, Mass., February 20, 1922. He was born at St. Louis, April 25, 1869. He graduated at the Washington University with the degree of M.Sc. in 1891, took the degree of A.M. at

Harvard in 1896, and the degree of Ph.D. at Leipzig in 1898. He was instructor at Smith Academy, St. Louis, 1891–1894, and at Washington University, 1893–1894. He went to Harvard as instructor in 1898, became assistant professor in 1904 and associate professor in 1914. He was one of the editors of the *Bulletin of the American Mathematical Society*, 1900–1902, and of the *Transactions*, 1902–1910. He contributed several articles to the *Annals of Mathematics* and to other periodicals, the most important being "Invariants of the general linear differential equation and their relation to the theory of continuous groups," *American Journal of Mathematics*, volume 21, 1899, pages 25–84, and "Examples of the construction of Riemann's surfaces for the inverse of rational functions, by the method of conformal representation," *Annals of Mathematics*, volume 12, 1898–1899, pages 1–26. The former was his dissertation under Lie at Leipzig.

GEORGE BRUCE HALSTED died in New York, March 19, 1922. Professor Halsted was born in Newark, N. J., November 25, 1853. He received the degree of A.B. from Princeton in 1875, and in 1879 the degree of Ph.D. from Johns Hopkins, where he was fellow, 1876–1878. He was instructor at Princeton, 1878–1881, and professor at the University of Texas, 1884–1903, at St. Johns College, Md., 1903, at Kenyon College, Ohio, 1903–1906, and at the Colorado State Teachers College, 1906–1912.

His first book was *Mensuration*, Boston, 1881. In New York, appeared *Elements of Geometry*, 1885; *Elementary Synthetic Geometry*, 1892; "Projective Geometry," Chapter II in *Higher Mathematics*, edited by Mansfield Merriman and R. S. Woodward, 2d ed., 1898, published separately as No. 2 of *Mathematical Monographs*, 1906; *Rational Geometry*, 1904, revised 1907 (translated into French by P. Barbarin, Paris, 1911).

His most important work was the translation of writings on non-Euclidean geometry. Lobachevski's *Researches on the Theory of Parallels* and Bolyai's *Science Absolute of Space* were first published at Austin, Texas, in 1891, as parts of the "Neomonic Series." They are now published in Chicago. His translation of Saccheri, *Euclides Vindicatus*, as *Euclid freed from every Fleck*, first appeared (all but the last thirteen pages) in this MONTHLY, *1894–1898*. He also translated some of Poincaré's writings on the foundations of science; namely, *Science and Hypothesis*, *The Value of Science*, and *Science and Method*, published in a single volume in 1913, New York and Garrison, N. Y.

Professor Halsted wrote numerous articles, biographical sketches, criticisms, etc., which are scattered through this MONTHLY, *Science* and other periodicals. Sommerville gives a list of ninety titles in his *Bibliography of Non-Euclidean Geometry* (London, 1911).

The honorary degree of doctor of science has been conferred on Sir THOMAS MUIR by the University of Cape Town, in recognition of his researches in mathematics and the history of mathematics.

Professor E. I. FREDHOLM, of the University of Stockholm, has been elected correspondent of the Paris Academy of Sciences in the section of geometry, as successor to the late Professor H. A. Schwarz.

A colloquium on the fundamental concepts of electrodynamics and of the electron theory of matter was held at the University of Wisconsin on March 30, 31, and April 1, 1922. The particular occasion for this meeting was the presence of Professor H. A. LORENTZ, the founder of the electron theory. The majority of those present were from the universities and colleges of the middle west, although both the Atlantic and Pacific coasts were represented. During the week preceding the colloquium proper, Dr. Lorentz gave four lectures on the general subject of light and the constitution of matter. These lectures, attended by a large and enthusiastic group of students and physicists, began with the basic concepts of the electromagnetic field, and traced briefly the developments which have led to the modern viewpoint. Professor Lorentz considered the successes and logical difficulties of the Bohr-atom theory, as extended by Sommerfeld and others, and discussed at some length the Michelson-Morley experiment and restricted relativity. In the last lecture a quantum-theory explanation of the Zeemann effect was given, to replace the older theory, based on classical electrodynamics.

During the colloquium itself the following lectures were given: "The experimental basis for the laws of electrodynamic action" by W. F. G. SWANN; "Deduction of the laws of electrodynamics from the relativity principle" by LEIGH PAGE; "Analytic formulation of electromagnetic theory through the field concept" by MAX MASON; "The structure of the electron" by D. L. WEBSTER; "The rotating earth as a reference system for light propagation" by L. SILBERSTEIN; "Application of statistical mechanics to electron theory" by A. C. LUNN; "Scattering of light and resonance radiation in relation to optical theories" by R. W. WOOD; "Thermal radiation—A discussion of recent experimental results" by C. E. MENDENHALL; "Electron theory of metals, volume phenomena" by P. W. BRIDGEMAN; "Electron theory of metals, surface phenomena" by P. T. COMPTON. At the conclusion of each paper the discussion was opened by Dr. Lorentz. The searching keenness, kindly interest, and revealing inspiration of his criticism will undoubtedly stimulate the scientific activities of all those who were present.

Published October 15, 1922.

THE APRIL MEETING OF THE KENTUCKY SECTION.

The sixth regular annnal meeting of the Kentucky Section was held in the Physics Building, Georgetown College, Georgetown, Kentucky, on April 8, 1922. Professor C. H. Richardson, chairman, presided at both the morning and the afternoon sessions.

There were nineteen in attendance, including the following eight members of the Association:

R. V. Blair, P. P. Boyd, W. J. Brezler, J. M. Davis, H. H. Downing, Elizabeth LeStourgeon, C. H. Richardson, G. A. Seubert.

Those attending the meeting were entertained at luncheon by Professor C. H. Richardson. At the business meeting which followed the presentation of papers, Professor ELIZABETH LESTOURGEON was elected chairman and Professor C. H. RICHARDSON, secretary-treasurer. Upon the invitation of Dean Boyd, it was voted to hold the next meeting at the University of Kentucky.

The following papers were presented:

(1) "Invariants in the singular case of quadrics" by Professor H. H. DOWNING;

(2) "Infinite series in the theory of potential" by Professor ELIZABETH LESTOURGEON;

(3) "Some proofs of Pascal's theorem" by Mr. R. V. BLAIR;

(4) "The mystic hexagram configuration" by Dean P. P. BOYD;

(5) "Kramp's faculty function" by Professor C. H. RICHARDSON.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. In this paper Professor Downing considered two quadratic forms $\varphi = \Sigma_1^4 a_{ij}x_ix_j$ and $\psi = \Sigma_1^4 b_{ij}x_ix_j$, each of which is singular, the pencil of the pair, $\varphi - \lambda\psi$, having a determinant identically zero in λ. It was shown that the forms are completely classified by means of their ranks and the elementary divisors of the pencil. According to the ranks there are nine cases. These divide into thirty sub-cases according to the elementary divisors. The elementary divisors were found for each of the thirty sub-cases.

2. In the theory of logarithmic potential the distribution of mass along the boundary of a circle has been studied, and an expression for the density, in the form of a Fourier Series, has been obtained by a limiting process out of the series for the potential. Professor LeStourgeon made a similar study in the case of certain other analytic curves in the plane, both for the logarithmic potential and for the conjugate potential function.

3. After stating Pascal's theorem, Mr. Blair gave two proofs from Steiner (*Vorlesungen über Synthetische Geometrie*, volume 1, Leipzig, 1887, p. 16), one from Salmon (*A Treatise on Conic Sections*, London, 1879, p. 245), and the ordinary proof of projective geometry (cf. O. Veblen and J. W. Young, *Projective*